Periodic geodesics on translation surfaces

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1   Introduction

Let M be a compact connected oriented surface. The surface M is called a *translation surface* if it is equipped with a *translation structure*, that is, an atlas of charts such that all transition functions are translations in $\mathbb{R}^2$. It is assumed that the chart domains cover all surface M except for finitely many points called *singular*. The translation structure induces the structure of a smooth manifold, a flat Riemannian metric, and a Borel measure on the surface M punctured at the singular points. We require that the metric has a cone type singularity at each singular point; then the area of the surface is finite. The cone angle is of the form $2\pi m$, where m is an integer called the *multiplicity* of the singular point. A singular point of multiplicity 1 is called *removable*; it is rather a marked point than a true singularity of the metric.

Furthermore, the translation structure allows us to identify the tangent space at any nonsingular point $x \in M$ with the Euclidean space $\mathbb{R}^2$. In particular, the unit tangent space at any point is identified with the unit circle $S^1 = \{v \in \mathbb{R}^2 : |v| = 1\}$. The velocity is an integral of the geodesic flow with respect to this identification. Thus each oriented geodesic has a *direction*, which is a uniquely determined vector in S^1. The direction of an unoriented geodesic is determined up to multiplying by ± 1.

Suppose X is a Riemann surface (one-dimensional complex manifold) homeomorphic to the surface M. Any nonzero Abelian differential on X defines a translation structure on M. The zeroes of the differential are singular points of the translation structure, namely, a zero of order k is a singular point of multiplicity $k + 1$. Every translation structure without removable singular points can be obtained this way.

Any geodesic joining a nonsingular point to itself is *periodic* (or *closed*). We regard periodic geodesics as simple closed unoriented curves. Any periodic geodesic is included in a family of freely homotopic periodic geodesics of the same length and direction. The geodesics of the family fill an open connected domain. Unless the translation surface is a torus without singular points, this domain is an annulus. We call it a *cylinder* of periodic geodesics (or simply a *periodic cylinder*). A periodic cylinder is bounded by geodesic segments of

the same direction with endpoints at singular points. Such segments are called *saddle connections.*

The fundamental results on periodic geodesics of translation surfaces were obtained by Howard Masur in papers [M1], [M2], [M3]. These results can be summarized as follows.

Theorem 1.1 (Masur) *Let M be a translation surface without removable singular points.*

(a) There exists a periodic geodesic on M of length at most $\alpha\sqrt{S}$, where S is the area of M and $\alpha > 0$ is a constant depending only on the genus of M.

(b) The directions of periodic geodesics of M are dense in S^1.

(c) Let $N_1(M,T)$ denote the number of periodic cylinders of M of length at most $T > 0$. Then there exist $0 < c_1(M) < c_2(M) < \infty$ such that

$$c_1(M) \leq N_1(M,T)/T^2 \leq c_2(M)$$

for T sufficiently large.

The goal of the present paper is to prove effective versions of statements (a) and (c) of Theorem 1.1, and to generalize statement (b).

Throughout the paper we consider translation surfaces that have at least one singular point. There is no loss of generality as we can declare an arbitrary nonsingular point to be a removable singular point.

Our first result is an effective version of Theorem 1.1(a).

Theorem 1.2 *Let m be the sum of multiplicities of singular points of a translation surface M, and S be the area of M. Then there exists a periodic geodesic on M of length at most $\alpha_m\sqrt{S}$, where $\alpha_m = (8m)^{2^{3m-1}}$.*

It should be admitted that the proof of Theorem 1.1(a) given by Smillie in the survey [S] can be further developed to obtain an effective estimate of the constant α (unlike the proofs given in [M1] and [MT]). The techniques used below to prove Theorem 1.2 are very similar to those used in [S].

The periodic geodesic provided by Theorem 1.2 belongs to a cylinder of parallel periodic geodesics of the same length. Although the length of this cylinder is bounded, its width, in general, may be arbitrarily small. Nevertheless it is possible to find a periodic cylinder whose area is not very small compared to the area of the whole surface.

Theorem 1.3 *Let m be the sum of multiplicities of singular points of a translation surface M, and S be the area of M. Then there exists a cylinder of periodic geodesics of length at most $\beta_m\sqrt{S}$, where $\beta_m = 2^{2^{4m}}$, and of area at least S/m.*

The following theorem shows, in particular, that almost every point of a translation surface lies on a periodic geodesic.

Theorem 1.4 *Let m be the sum of multiplicities of singular points of a translation surface M, and S be the area of M. For any $\delta \in (0, 1)$ there exist pairwise disjoint periodic cylinders $\Lambda_1, \ldots, \Lambda_k$ of length at most $(8m\delta^{-1})^{2^{3m-1}}\sqrt{S}$ such that the area of the union $\Lambda_1 \cup \ldots \cup \Lambda_k$ is at least $(1 - \delta)S$.*

The group $\mathrm{SL}(2, \mathbb{R})$ acts on the set of translation structures on a given surface by postcomposition of the chart maps with linear transformations from $\mathrm{SL}(2, \mathbb{R})$. This action preserves singular points along with their multiplicities, geodesics, and the measure induced by translation structure. It does not preserve directions and lengths of geodesic segments however. This observation allows one to derive statement (b) of Theorem 1.1 from statement (a). In the same way Theorem 1.3 leads to the following result.

Theorem 1.5 *Let m be the sum of multiplicities of singular points of a translation surface M, and S be the area of M. Then the directions of periodic cylinders of area at least S/m are dense in S^1.*

A plane polygon is called *rational* if the angle between any two of its sides is a rational multiple of π. A construction of Zemlyakov and Katok [ZK] associates to any rational polygon Q a translation surface M so that the study of the billiard flow in Q can be reduced to the study of the geodesic flow on M. In view of this construction, Theorem 1.1(b) implies that directions of periodic billiard orbits in Q are dense in the set of all directions. Boshernitzan, Galperin, Krüger, and Troubetzkoy [BGKT] strengthened this result.

Theorem 1.6 ([BGKT]) *For any rational polygon Q, the periodic points of the billiard flow in Q are dense in the phase space of the flow. Moreover, there exists a dense G_δ-set $Q_0 \subset Q$ such that for every point $x \in Q_0$ the directions of periodic billiard orbits starting at x form a dense subset of S^1.*

An analogous result for translation surfaces—periodic points of the geodesic flow are dense in the phase space of the flow—can be obtained in the same way (cf. [MT]). In this paper we prove a further strengthening of Theorem 1.1(b).

Theorem 1.7 *(a) For any translation surface M, there exists a G_δ-set $M_0 \subset M$ of full measure such that for every point $x \in M_0$ the directions of periodic geodesics passing through x form a dense subset of S^1.*

(b) For any rational polygon Q, there exists a G_δ-subset $Q_0 \subset Q$ of full measure such that for every point $x \in Q_0$ the directions of periodic billiard orbits starting at x form a dense subset of S^1.

Our final result is an effective version of Theorem 1.1(c).

Theorem 1.8 *Let M be a translation surface. Denote by $N_1(M,T)$ the number of cylinders of periodic geodesics on M of length at most $T > 0$. By $N_2(M,T)$ denote the sum of areas of these cylinders. Then*

$$\left((600m)^{(2m)^{2m}}\right)^{-1} s^2 S^{-2} T^2 \leq N_2(M,T)/S \leq$$

$$N_1(M,T) \leq (400m)^{(2m)^{2m}} s^{-2} T^2$$

for any $T \geq 2^{2^{4m}} \sqrt{S}$, where S is the area of M, m is the sum of multiplicities of singular points of M, and s is the length of the shortest saddle connection of M.

The paper is organized as follows. Section 2 contains definitions, notation, and preliminaries. The results on existence of periodic geodesics (Theorems 1.2, 1.3, and 1.4) are obtained in Section 3. The results on density of directions of periodic geodesics (Theorems 1.5 and 1.7) are obtained in Section 4. Section 5 is devoted to the proof of Theorem 1.8.

2   Preliminaries

Let M be a compact connected oriented surface. A *translation structure* on M is an atlas of coordinate charts $\omega = \{(U_\alpha, f_\alpha)\}_{\alpha \in \mathcal{A}}$, where U_α is a domain in M and f_α is a homeomorphism of U_α onto a domain in $\mathbb{R}^2$, such that:

• all transition functions are translations in $\mathbb{R}^2$;

• chart domains U_α, $\alpha \in \mathcal{A}$, cover all surface M except for finitely many points (called *singular* points);

• the atlas ω is maximal relative to the two preceding conditions;

• a punctured neighborhood of any singular point covers a punctured neighborhood of a point in $\mathbb{R}^2$ via an m-to-1 map which is a translation in coordinates of the atlas ω; the number m is called the *multiplicity* of the singular point.

A *translation surface* is a compact connected oriented surface equipped with a translation structure.

The translation structures are also (and probably better) known as "orientable flat structures" or "admissible positive F-structures".

Let M be a translation surface and ω be the translation structure of M. Each translation of the plane $\mathbb{R}^2$ is a smooth map preserving orientation, Euclidean metric and Lebesgue measure on $\mathbb{R}^2$. Hence the translation structure ω induces a smooth structure, an orientation, a flat Riemannian metric, and a finite Borel measure on the surface M punctured at the singular points of ω. Each singular point of ω is a cone type singularity of the metric. The cone angle

is equal to $2\pi m$, where m is the multiplicity of the singular point. Any geodesic of the metric is a straight line in coordinates of the atlas ω. A geodesic hitting a singular point is considered to be singular, its further continuation is undefined. Almost every element of the tangent bundle gives rise to a nonsingular geodesic.

The translation structure ω allows us to identify the tangent space at any nonsingular point $x \in M$ with the Euclidean space $\mathbb{R}^2$. In particular, the unit tangent space at any point is identified with the unit circle $S^1 = \{v \in \mathbb{R}^2 : |v| = 1\}$. The velocity is an integral of the geodesic flow with respect to this identification. Thus each oriented geodesic is assigned a *direction* $v \in S^1$. The direction of an unoriented geodesic is determined up to multiplying by ± 1. For any $v \in S^1$, let M_v denote the invariant surface of the phase space of the geodesic flow corresponding to the motion with velocity v. Clearly, the restriction of the geodesic flow to M_v can be regarded as a flow on the surface M. This flow is called the *directional flow* in direction v. If a point $x \in M$ is singular or at least one of geodesics starting at x in the directions $\pm v$ hits a singular point, then the directional flow is only partially defined at the point x. The directional flow is fully defined on a subset of full measure (depending on v) and preserves the measure on M.

A singular point of multiplicity 1 is called *removable*. If $x \in M$ is a removable singular point of the translation structure ω, then there exists a translation structure $\omega_+ \supset \omega$ such that x is not a singular point of ω_+. On the other hand, let x be a nonsingular point of M. Suppose ω_- is the set of charts $(U_\alpha, f_\alpha) \in \omega$ such that $x \notin U_\alpha$. Then ω_- is a translation structure on M and x is a removable singular point of ω_-.

Let p be the genus of a translation surface M, k be the number of singular points of M, and m be the sum of multiplicities of the singular points. Then $m = 2p - 2 + k$. It follows that there are no translation structures on the sphere, a translation torus can have only removable singular points, and a translation surface of genus $p > 1$ has at least one nonremovable singular point.

Suppose X is a complex structure on a compact connected oriented surface M. Let q be a nonzero Abelian differential (holomorphic 1-form) on X. A chart (U, z), where U is a domain in M and z is a homeomorphism of U onto a domain in $\mathbb{C}$, is called a *natural parameter* of the differential q if $q = dz$ in U with respect to the complex structure X. Let ω denote the atlas of all natural parameters of q. The natural identification of $\mathbb{C}$ with $\mathbb{R}^2$ allows us to consider ω as an atlas of charts ranging in $\mathbb{R}^2$. It is easy to observe that ω is a translation structure on M. The singular points of ω are the zeroes of the differential q, namely, a zero of order n is a singular point of multiplicity $n + 1$. Each translation structure on M without removable singular points can be obtained by this construction.

Another way to construct translation surfaces is to glue them from poly-

gons. Let $Q_1, \ldots, Q_n$ be disjoint plane polygons. The natural orientation of $\mathbb{R}^2$ induces an orientation of the boundary of every polygon. Suppose all sides of the polygons $Q_1, \ldots, Q_n$ are groupped in pairs such that two sides in each pair are of the same length and direction, and of opposite orientations. Glue the sides in each pair by translation. Then the union of the polygons $Q_1, \ldots, Q_n$ becomes a surface M. By construction, the surface M is compact and oriented. Suppose M is connected (if it is not, then we should apply the construction to a smaller set of polygons). The restrictions of the identity map on $\mathbb{R}^2$ to the interiors of the polygons $Q_1, \ldots, Q_n$ can be regarded as charts of M. This finite collection of charts extends to a translation structure ω on M. The translation structure ω is uniquely determined if we require that the set of singular points of ω be the set of points corresponding to vertices of the polygons $Q_1, \ldots, Q_n$.

A particular case of the latter construction is the so-called Zemlyakov-Katok construction, which descends from the paper [ZK]. Let Q be a plane polygon. Let $s_1, \ldots, s_n$ be sides of Q. For any i, $1 \leq i \leq n$, let $\tilde{r}_i$ denote the reflection of the plane in the side s_i and r_i denote the linear part of $\tilde{r}_i$. By $G(Q)$ denote the subgroup of $O(2)$ generated by the reflections $r_1, \ldots, r_n$. The polygon Q is called *rational* if the group $G(Q)$ is finite. All angles of a rational polygon are rational multiples of π. This property is equivalent to being rational provided the polygon is simply connected. Suppose the polygon Q is rational. Let Q_g, $g \in G(Q)$, be disjoint polygons such that for any $g \in G(Q)$ there exists an isometry $R_g : Q_g \to Q$ with linear part g. Now for any $i \in \{1, \ldots, n\}$ and any $g \in G(Q)$ glue the side $R_g^{-1} s_i$ of the polygon Q_g to the side $R_{gr_i}^{-1} s_i$ of the polygon Q_{gr_i} by translation. This transforms the union of polygons $Q_1, \ldots, Q_n$ into a compact connected oriented surface M. Observe that the collection of isometries R_g, $g \in G(Q)$, gives rise to a continuous map $f_Q : M \to Q$. The surface M is endowed with a translation structure ω as described above. Singular points of ω correspond to vertices of the polygon Q. Namely, the vertex of any angle of the form $2\pi n_1/n_2$, where n_1 and n_2 are coprime integers, gives rise to N/n_2 singular points of multiplicity n_1, where N is the cardinality of the group $G(Q)$.

The Zemlyakov-Katok construction is crucial for the study of the billiard flow in rational polygons. The *billiard flow* in a polygon Q is a dynamical system that describes a point-like mass moving freely within the polygon Q subject to elastic reflections in the boundary of Q. A *billiard orbit* in Q is a broken line changing its direction at interior points of the sides of Q according to the law "the angle of incidence is equal to the angle of reflection". A billiard orbit hitting a vertex of the polygon Q is supposed to stop at this vertex. A billiard orbit starting at a point $x \in Q$ in a direction $v \in S^1$ is *periodic* if it returns eventually to the point x in the direction v. Suppose Q is a rational polygon. Let M be the translation surface associated to Q. Let $f_Q : M \to Q$ be the continuous map introduced above. It is easy to see that f_Q maps any

geodesic on the surface M onto a billiard orbit in Q. Conversely, any billiard orbit in Q is the image of a (not uniquely determined) geodesic on M. By construction, there exists a domain $D \subset M$ such that f_Q maps the domain D isometrically onto the interior of the polygon Q and, moreover, the chart $(D, f_Q|D)$ is an element of the translation structure of M. If L is a geodesic starting at a point $x \in D$ in a direction $v \in S^1$, then $f_Q(L)$ is the billiard orbit in Q starting at the point $f_Q(x)$ in the same direction.

Let M be a translation surface. A domain $U \subset M$ containing no singular points is called a *triangle* (a *polygon*, an *n-gon*) if it is isometric to the interior of a triangle (resp. a polygon, an n-gon) in the plane $\mathbb{R}^2$. Suppose $h : U \to P \subset \mathbb{R}^2$ is a corresponding isometry. The inverse map $h^{-1} : P \to U$ can be extended to a continuous map of the closure of P to M. The images of sides and vertices of the polygon P under this extension are called *sides* and *vertices* of U. Every side of U is either a geodesic segment or a union of several parallel segments separated by singular points. Note that the number of vertices of the n-gon U may be less than n. A *triangulation* of the translation surface M is its partition into a finite number of triangles.

A *saddle connection* is a geodesic segment joining two singular points or a singular point to itself and having no singular points in its interior (note that singular points are saddles for directional flows). Two saddle connections of a translation surface are said to be *disjoint* if they have no common interior points (common endpoints are allowed). Three saddle connections are pairwise disjoint whenever they are sides of a triangle. For any $T > 0$ there are only finitely many saddle connections of length at most T. In particular, there exists the shortest saddle connection (probably not unique).

The following proposition is well known (see, e.g. [MT], [Vo]).

Proposition 2.1 *(a) Any collection of pairwise disjoint saddle connections can be extended to a maximal collection.*

(b) Any maximal collection of pairwise disjoint saddle connections forms a triangulation of the surface M such that all sides of each triangle are saddle connections.

(c) For any maximal collection, the number of saddle connections is equal to $3m$, and the number of triangles in the corresponding triangulation is equal to $2m$, where m is the sum of multiplicities of singular points.

Any geodesic joining a nonsingular point to itself is called *periodic* (or *closed*); such a geodesic is a periodic orbit of a directional flow. We only consider *primitive* periodic geodesics, that is, the period of the geodesic is its length. Also, we regard periodic geodesics as unoriented curves. If a geodesic starting at a point $x \in M$ is periodic, then all geodesics starting at nearby points in the same direction are also periodic. Actually, each periodic geodesic belongs to a family of freely homotopic periodic geodesics of the same length

and direction. If M is a torus without singular points, then this family fills the whole surface M. Otherwise the family fills a domain homeomorphic to an annulus. This domain is called a *cylinder* of periodic geodesics (or simply a *periodic cylinder*) since it is isometric to a cylinder $\mathbb{R}/l\mathbb{Z} \times (0, w)$, where $l, w > 0$. The numbers l and w are called the *length* and the *width* of the periodic cylinder. The cylinder is bounded by saddle connections of the same direction.

Let $\omega = \{(U_\alpha, f_\alpha)\}_{\alpha \in \mathcal{A}}$ be a translation structure on the surface M. For any linear operator $a \in \mathrm{SL}(2, \mathbb{R})$ the atlas $\{(U_\alpha, a \circ f_\alpha)\}_{\alpha \in \mathcal{A}}$ is also a translation structure on M. We denote this structure by $a\omega$. Clearly, $(a_1 a_2)\omega = a_1(a_2\omega)$ for any $a_1, a_2 \in \mathrm{SL}(2, \mathbb{R})$ so we have an action of the group $\mathrm{SL}(2, \mathbb{R})$ on the set of translation structures on M. The translation structures ω and $a\omega$ share the same singular points of the same multiplicities and the same geodesics. In addition, they induce the same measure on the surface M.

To each oriented geodesic segment L of the translation structure ω we associate the vector $v \in \mathbb{R}^2$ of the same length and direction. If the segment L is not oriented, then the vector v is determined up to reversing its direction. For any $a \in \mathrm{SL}(2, \mathbb{R})$ the vector av is associated to L with respect to the translation structure $a\omega$. Given a direction $v_1 \in S^1$, the length of the orthogonal projection of v on the direction v_1 is called the *projection* of the segment L on v_1 (with respect to ω).

3   Existence of periodic geodesics

To prove Theorems 1.2, 1.3, and 1.4, we need the following proposition.

Proposition 3.1 *Let M be a translation surface of area S. Suppose $L_1, \ldots, L_k$ ($k \geq 0$) are pairwise disjoint saddle connections of length at most $\sqrt{2S}$. Then at least one of the following possibilities occur:*

(1) saddle connections $L_1, \ldots, L_k$ partition the surface into finitely many domains such that each domain is either a periodic cylinder of length at most $\sqrt{S}$ or a triangle bounded by three saddle connections;

(2) there exists a saddle connection L of length at most $2\sqrt{2S}$ disjoint from $L_1, \ldots, L_k$.

Proof.   First consider the case when a small neighborhood of some singular point x_0 contains an open sector K of angle π that is disjoint from saddle connections $L_1, \ldots, L_k$. Suppose there exists a geodesic segment J of length at most $\sqrt{2S}$ that goes out of the point x_0 across sector K and ends in a point y which is either a singular point or an interior point of some saddle connection L_j, $1 \leq j \leq k$. We can assume without loss of generality that the interior of the segment J contains no singular point and is disjoint from saddle connections $L_1, \ldots, L_k$. If y is a singular point, then J is a saddle connection disjoint from

$L_1, \ldots, L_k$, thus condition (2) holds. Suppose y is an interior point of L_j. Let K' be an open sector with vertex at the point x_0 crossed by the segment J. We assume that each geodesic I going out of x_0 across the sector K' intersects L_j before this geodesic hits a singular point or intersects another given saddle connection. This condition holds, for instance, when the angle of the sector K' is small enough. Let I_0 denote the segment of the geodesic I from the point x_0 to the first intersection with L_j. The segments I_0, J, and a subsegment of L_j are sides of a triangle, hence the length of I_0 is less than the sum of lengths of J and L_j, which, in turn, is at most $2\sqrt{2S}$. Without loss of generality it can be assumed that K' is the maximal sector with the above property. By the maximality, both geodesics going out of x_0 along the boundary of K' hit singular points before they intersect any of the given saddle connections. It follows that these geodesics are saddle connections of length at most $2\sqrt{2S}$. By construction, any of the two saddle connections either is disjoint from saddle connections $L_1, \ldots, L_k$ or coincides with one of them. Since the angle of the sector K' is less than π, at least one of the boundary saddle connections crosses the sector K; such a saddle connection is not among $L_1, \ldots, L_k$. Thus condition (2) holds.

Now suppose that any geodesic segment of length $\sqrt{2S}$ going out of the point x_0 across sector K does not reach a singular point and does not intersect saddle connections $L_1, \ldots, L_k$. Let I be the geodesic segment of length $\sqrt{S}$ that goes out of the singular point x_0 dividing the sector K into two equal parts. By v denote one of two directions orthogonal to the direction of I. Let $\{F^t\}_{t \in \mathbb{R}}$ be the directional flow in direction v. By I_0 denote the segment I without its endpoints. For any $t > 0$ let D_t denote the set of points of the form $F^\tau x$, where $x \in I_0$ and $0 < \tau < t$. If t is small enough, then D_t is a rectangle with sides $\sqrt{S}$ and t. Let t_1 be the maximal number with this property. The area of the rectangle D_{t_1} is equal to $t_1\sqrt{S}$, hence $t_1 \leq \sqrt{S}$. Set $D'_{t_1} = D_{t_1} \cup F^{t_1}I_0$. Any point $x \in D'_{t_1}$ can be joined to the point x_0 by a geodesic segment J_x such that all interior points of J_x are contained in D_{t_1}. The segment J_x crosses the sector K and the length of J_x is at most $\sqrt{t_1^2 + S} \leq \sqrt{2S}$. It follows that the set D'_{t_1} is disjoint from saddle connections $L_1, \ldots, L_k$ and contains no singular points. Likewise, the set $D'_{-t_1} = \{F^t x \mid x \in I_0, \ -t_1 \leq t < 0\}$ is also disjoint from $L_1, \ldots, L_k$ and contains no singular points. By the choice of t_1, there exists a point $x_1 \in I_0$ such that the point $F^{t_1}x_1$ either is singular or belongs to I_0. As $F^{t_1}x_1 \in D'_{t_1}$, we have $F^{t_1}x_1 \in I_0$. Let y_0 denote the endpoint of I different from x_0. It is easy to see that the set $I_1 = I_0 \cap F^{t_1}I_0$ is an open subsegment of I_0 and y_0 is an endpoint of I_1. Suppose that $I_1 \neq I_0$. Let y be an endpoint of I_1 that is an interior point of I. Obviously, $F^{-t_1}y \in I$. Since y is an endpoint of I_1, it follows that $F^{-t_1}y = x_0$. On the other hand, $F^{-t_1}y \in D'_{-t_1}$. This contradiction proves that $I_1 = I_0$. It follows that the set D'_{t_1} is a union of periodic geodesics of length t_1 and of direction v. Therefore D'_{t_1} is contained in a periodic cylinder

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Λ of length $t_1 \le \sqrt{S}$. The cylinder Λ contains the sector K. By construction, at least some of periodic geodesics in Λ do not intersect saddle connections $L_1, \dots, L_k$. It follows easily that the whole cylinder is disjoint from $L_1, \dots, L_k$.

Now suppose that condition (2) does not hold. We have to show that condition (1) does hold in this case. By the above the saddle connections $L_1, \dots, L_k$ divide a small neighborhood of each singular point into sectors of angle at most π. Moreover, each sector of angle π is contained within a periodic cylinder of length at most $\sqrt{S}$ disjoint from $L_1, \dots, L_k$. The saddle connections $L_1, \dots, L_k$ partition the surface M into finitely many domains. Let D be one of these domains. Take a singular point x_0 at the boundary of D. A small neighborhood of x_0 intersects the domain D in one or more sectors of angle at most π. Let K be one of such sectors. If the sector K is of angle π, then it is contained in a periodic cylinder Λ of length at most $\sqrt{S}$ disjoint from $L_1, \dots, L_k$. Clearly, $\Lambda \subset D$. The lengths of saddle connections bounding the cylinder Λ do not exceed the length of Λ. Any of these saddle connections either is disjoint from $L_1, \dots, L_k$ or coincides with one of them. Since condition (2) does not hold, all saddle connections bounding Λ are among $L_1, \dots, L_k$. This means that $D = \Lambda$. Now consider the case when the angle of the sector K is less than π. The sector K is bounded by some saddle connections L_i and L_j. Let T be a triangle such that T contains the sector K, the saddle connection L_i is a side of T, and a subsegment of L_j is another side of T. Obviously, $T \subset D$. We can assume without loss of generality that T is the maximal triangle with this property. By L_0 denote the side of T different from L_i and from the subsegment of L_j. Let J be a geodesic segment that goes out of the point x_0 across sector K, crosses the triangle T, and ends in a point $y \in L_0$. The length of J is less than the sum of lengths of L_i and L_j, which, in turn, is at most $2\sqrt{2S}$. Since condition (2) does not hold, the point y can not be singular. It follows that the side L_0 is a single geodesic segment. By the maximality of T, the whole saddle connection L_j is a side of T. Then L_0 is a saddle connection. By the triangle inequality, the length of L_0 is at most $2\sqrt{2S}$. Hence L_0 is one of the saddle connections $L_1, \dots, L_k$ as otherwise L_0 is disjoint from $L_1, \dots, L_k$. This means that $D = T$. Thus condition (1) holds. ∎

For any operator $a \in \mathrm{SL}(2, \mathbb{R})$, let $\|a\|$ denote the *norm* of a and $C(a)$ denote the *condition number* of a:

$$\|a\| = \max_{v \in \mathbb{R}^2, \, |v|=1} |av|, \qquad C(a) = \max\left(\|a\|, \|a^{-1}\|\right).$$

Obviously, $C(a_1 a_2) \le C(a_1) C(a_2)$ for any $a_1, a_2 \in \mathrm{SL}(2, \mathbb{R})$. Suppose L is a geodesic segment of a translation structure ω. Then the lengths of the segment L with respect to translation structures ω and $a\omega$ differ by at most $C(a)$ times.

Proof of Theorem 1.4. Let $\delta \in (0, 1)$. Set $\varepsilon = (8m\delta^{-1})^{-2^{3m-1}}$. Note that $\varepsilon^{2^{-3m}} \le 1/2$. Let ω denote the translation structure of the translation surface

M. Suppose that a sequence $L_1, \ldots, L_k$ ($k \geq 0$) of pairwise disjoint saddle connections and a sequence of operators $a_0, a_1, \ldots, a_k \in \mathrm{SL}(2, \mathbb{R})$ satisfy the following two conditions: (i) $C(a_i) \leq (1/\varepsilon)^{1-2^{-i}}$ for $i = 0, 1, \ldots, k$; and (ii) the length of L_i with respect to the translation structure $a_j\omega$, $1 \leq i \leq j \leq k$, does not exceed $2\sqrt{2S}\,\varepsilon^{2^{-j}}$. Furthermore, suppose there exists a saddle connection L disjoint from $L_1, \ldots, L_k$ and of length at most $2\sqrt{2S}$ with respect to the translation structure $a_k\omega$. Let $v, u \in S^1$ be orthogonal vectors such that v is parallel to the saddle connection L with respect to $a_k\omega$. Define an operator $b \in \mathrm{SL}(2, \mathbb{R})$ by equalities $bv = \varepsilon^{2^{-k-1}} v$, $bu = \varepsilon^{-2^{-k-1}} u$. Further, set $a_{k+1} = ba_k$. Obviously, $C(b) = (1/\varepsilon)^{2^{-k-1}}$, hence $C(a_{k+1}) \leq C(b)C(a_k) \leq (1/\varepsilon)^{2^{-k-1}}(1/\varepsilon)^{1-2^{-k}} = (1/\varepsilon)^{1-2^{-k-1}}$. The length of the saddle connection L with respect to $a_{k+1}\omega$ is at most $2\sqrt{2S}\,\varepsilon^{2^{-k-1}}$, while the length of saddle connections $L_1, \ldots, L_k$ with respect to $a_{k+1}\omega$ is at most $2\sqrt{2S}\,\varepsilon^{2^{-k}}C(b) = 2\sqrt{2S}\,\varepsilon^{2^{-k-1}}$. Thus the sequence of saddle connections $L_1, \ldots, L_k, L$ and the sequence of operators $a_0, a_1, \ldots, a_k, a_{k+1}$ satisfy the conditions (i) and (ii).

Pairs of sequences satisfying conditions (i) and (ii) do exist, for example, the empty sequence of saddle connections and the sequence consisting of one operator $a_0 = 1$. By Proposition 2.1, the number of pairwise disjoint saddle connections can not exceed $3m$. Therefore there exists a pair of sequences $L_1, \ldots, L_k$ and $a_0, a_1, \ldots, a_k$ satisfying conditions (i) and (ii) with maximal possible k. The lengths of the saddle connections $L_1, \ldots, L_k$ with respect to $a_k\omega$ are at most $2\sqrt{2S}\varepsilon^{2^{-k}} \leq 2\sqrt{2S}\varepsilon^{2^{-3m}} \leq \sqrt{2S}$, thus Proposition 3.1 applies. By the maximality of k, there is no saddle connection disjoint from $L_1, \ldots, L_k$ and of length at most $2\sqrt{2S}$ with respect to the translation structure $a_k\omega$. Thus the saddle connections $L_1, \ldots, L_k$ partition the surface M into finitely many domains such that each domain is either a periodic cylinder of length at most $\sqrt{S}$ with respect to $a_k\omega$ or a triangle bounded by three saddle connections. Any triangle in this partition is of area at most $\frac{1}{2}(2\sqrt{2S}\,\varepsilon^{2^{-k}})^2$ with respect to both $a_k\omega$ and ω. It follows from Proposition 2.1 that there are at most $2m$ triangles in the partition. Hence the union of these triangles is of area at most

$$m(2\sqrt{2S}\,\varepsilon^{2^{-k}})^2 \leq m(2\sqrt{2S}\,\varepsilon^{2^{-3m}})^2 = \delta S.$$

Then the union of all periodic cylinders in the partition is of area at least $(1-\delta)S$. It remains to observe that the length of each periodic cylinder with respect to the translation structure ω is at most $C(a_k)\sqrt{S} \leq (1/\varepsilon)^{1-2^{-k}}\sqrt{S} < \varepsilon^{-1}\sqrt{S} = (8m\delta^{-1})^{2^{3m-1}}\sqrt{S}$. ∎

Proof of Theorem 1.2. By Theorem 1.4, for any $\delta \in (0, 1)$ the translation surface M admits a periodic geodesic of length at most $(8m\delta^{-1})^{2^{3m-1}}\sqrt{S}$. For any $T > 0$ the number of periodic cylinders of length at most T is finite, therefore there exists a shortest periodic geodesic. Let l denote its length. Since $l \leq (8m\delta^{-1})^{2^{3m-1}}\sqrt{S}$ for any $\delta \in (0, 1)$, we have $l \leq (8m)^{2^{3m-1}}\sqrt{S}$. ∎

Proof of Theorem 1.3. In the case $m = 1$ the translation surface M is a torus with one removable singular point. Here every cylinder of periodic geodesics fills the whole surface (up to the boundary saddle connection). As $\alpha_1 = 8^{2^2} = 2^{12} < 2^{2^4} = \beta_1$, the theorem follows from Theorem 1.2 in this case.

Consider the case $m \geq 2$. By Theorem 1.4, there exist pairwise disjoint periodic cylinders $\Lambda_1, \ldots, \Lambda_k$ of length at most $(8m^2)^{2^{3m-1}}\sqrt{S}$ such that the area of the union $\Lambda_1 \cup \ldots \cup \Lambda_k$ is at least $(1 - 1/m)S$. Each cylinder Λ_i can be triangulated by pairwise disjoint saddle connections. The number of triangles in any triangulation is at least 2. If we require that each side of any triangle is a saddle connection (not a union of several saddle connections), then the number of triangles is equal to the number of saddle connections bounding Λ_i, where saddle connections bounding Λ_i from both sides should be counted twice. All saddle connections used in triangulation of the cylinders $\Lambda_1, \ldots, \Lambda_k$ are pairwise disjoint since the cylinders are disjoint. By Proposition 2.1, we can add several saddle connections to obtain a partition of the surface M into $2m$ triangles bounded by disjoint saddle connections. It follows easily that the number k of cylinders is at most m. Moreover, if $k = m$ then the closure of the union $\Lambda_1 \cup \ldots \cup \Lambda_k$ is the whole surface M. In the latter case at least one of the cylinders $\Lambda_1, \ldots, \Lambda_k$ is of area not less than S/m. In the case $k < m$ one of the cylinders is of area not less than $(1 - 1/m)S/k \geq (1 - 1/m)S/(m - 1) = S/m$.

To complete the proof, it remains to show that $(8m^2)^{2^{3m-1}} \leq \beta_m$. It is easy to observe that $8m^2 \leq 2^{2m+1}$ and $2m+1 < 2^{m+1}$ for any integer $m \geq 1$. Hence, $(8m^2)^{2^{3m-1}} \leq 2^{(2m+1)2^{3m-1}} < 2^{2^{4m}} = \beta_m$. ∎

4 Density of directions of periodic geodesics

In this section we prove Theorems 1.5 and 1.7. They are derived from Theorems 1.3 and 1.4, respectively.

Proof of Theorem 1.5. Let ω denote the translation structure of the translation surface M. We have to show that for any direction $v \in S^1$ and any $\varepsilon > 0$ there exists a periodic cylinder of ω of area at least S/m such that the angle between v and the direction of the cylinder is less than ε.

Let $u \in S^1$ be a direction orthogonal to v. For any $\lambda > 1$ define an operator $a_\lambda \in \mathrm{SL}(2, \mathbb{R})$ by equalities $a_\lambda v = \lambda^{-1}v$, $a_\lambda u = \lambda u$. The sum of multiplicities of singular points of the translation structure $a_\lambda \omega$ is equal to m and the area of the surface M with respect to $a_\lambda \omega$ is equal to S. By Theorem 1.3, there exists a periodic cylinder Λ_λ of area at least S/m such that the length of Λ_λ with respect to $a_\lambda \omega$ is at most $l = 2^{2^{4m}}\sqrt{S}$. Let h_λ and w_λ be projections of a periodic geodesic from the cylinder Λ_λ on the directions v and u (with respect to the translation structure ω). Further, let φ_λ be the angle between v and the direction of Λ_λ, where the direction of the cylinder is chosen so that $0 \leq \varphi_\lambda \leq \pi/2$. Obviously, $h_\lambda \leq \lambda l$, $w_\lambda \leq \lambda^{-1}l$. Let s denote the length of the shortest

saddle connection of ω. Assuming λ is large enough, we have $w_\lambda \leq s/\sqrt{2}$. Since the length of the cylinder Λ_λ, which is equal to $\sqrt{h_\lambda^2 + w_\lambda^2}$, is not less than s, it follows that $h_\lambda \geq s/\sqrt{2}$. Then $\varphi_\lambda \leq \tan \varphi_\lambda = w_\lambda/h_\lambda \leq \lambda^{-1}l\sqrt{2}/s$, which tends to zero as λ goes to infinity. ∎

Proof of Theorem 1.7. Let M be a translation surface. Take a nonempty open subset U of the circle S^1. Let P_U denote the set of points $x \in M$ lying on periodic geodesics with directions in the set U. The set P_U is open. Let us show that this set is of full measure. Take a vector $v \in U$. Choose $\varepsilon > 0$ such that a direction $v' \in S^1$ belongs to U whenever the angle between v' and v is less than ε. Further, choose some $\delta \in (0,1)$. Let $u \in S^1$ be a direction orthogonal to v. For any $\lambda > 1$ define an operator $a_\lambda \in \mathrm{SL}(2,\mathbb{R})$ by equalities $a_\lambda v = \lambda^{-1}v$, $a_\lambda u = \lambda u$. Let ω denote the translation structure of M, m denote the sum of multiplicities of singular points of M, and S denote the area of M. By Theorem 1.4, there exist pairwise disjoint periodic cylinders $\Lambda_1, \ldots, \Lambda_k$ such that the length of every cylinder with respect to the translation structure $a_\lambda\omega$ is at most $l_\delta = (8m\delta^{-1})^{2^{3m-1}}\sqrt{S}$ and the area of the union $\Lambda_1 \cup \ldots \cup \Lambda_k$ is at least $(1-\delta)S$ (with respect to both $a_\lambda\omega$ and ω). Take some cylinder Λ_i. Let h and w be projections of a periodic geodesic from the cylinder Λ_i on the directions v and u (with respect to the translation structure ω). Further, let φ be the angle between v and the direction of Λ_i ($0 \leq \varphi \leq \pi/2$). Obviously, $h \leq \lambda l_\delta$, $w \leq \lambda^{-1}l_\delta$. Let s denote the length of the shortest saddle connection of ω. The length of the cylinder Λ_i is not less than s. If $\lambda \geq l_\delta\sqrt{2}/s$, then $w \leq s/\sqrt{2}$, hence $h \geq s/\sqrt{2}$. It follows that $\varphi \leq \tan\varphi = w/h \leq \lambda^{-1}l_\delta\sqrt{2}/s$. If, moreover, $\lambda > \varepsilon^{-1}l_\delta\sqrt{2}/s$, then $\varphi < \varepsilon$ and the direction of the cylinder Λ_i is in the set U. Thus the cylinders $\Lambda_1, \ldots, \Lambda_k$ are contained in the set P_U provided λ is sufficiently large. Then the area of P_U is at least $(1-\delta)S$. As δ can be chosen arbitrarily small, the area of P_U is equal to S.

Choose a sequence $U_1, U_2, \ldots$ of nonempty open subsets of the circle S^1 such that any other nonempty open subset of S^1 contains some U_i. By the above the sets $P_{U_1}, P_{U_2}, \ldots$ are open sets of full measure. Hence the set $P_\infty = \cap_{i=1}^\infty P_{U_i}$ is a G_δ-subset of full measure of the surface M. Take a point $x \in P_\infty$. For any positive integer i there exists a direction $v_i \in U_i$ that is the direction of a periodic geodesic passing through x. By construction, the sequence $v_1, v_2, \ldots$ is dense in S^1. The first statement of the theorem is proved.

To derive statement (b) of Theorem 1.7 from statement (a), we only need to recall the Zemlyakov-Katok construction (see Section 2). Let Q be a rational polygon and M be the translation surface associated to Q. By construction, there is a continuous map $f : M \to Q$ and a domain $D \subset M$ containing no singular points such that f maps the domain D isometrically onto the interior of the polygon Q. Moreover, if L is a geodesic passing through a point $x \in D$ in a direction $v \in S^1$, then $f(L)$ is the billiard orbit in Q starting at the point

$f(x)$ in the direction v. The billiard orbit $f(L)$ is periodic if and only if the geodesic L is periodic. By the above there exists a G_δ-set $M_0 \subset M$ of full measure such that for any $x \in M_0$ the directions of periodic geodesics passing through x are dense in S^1. Then the set $Q_0 = f(D \cap M_0)$ is a G_δ-subset of the polygon Q and the area of Q_0 is equal to the area of Q. For any point $x \in Q_0$ the directions of periodic billiard orbits in Q starting at x are dense in S^1. ∎

5 Lower quadratic estimates

Let M be a translation surface. In this section we obtain effective estimates of the growth functions $N_1(M, \cdot)$ and $N_2(M, \cdot)$, where $N_1(M, T)$ is the number of cylinders of periodic geodesics of length at most T and $N_2(M, T)$ is the sum of areas of those cylinders. Throughout the section m denotes the sum of multiplicities of singular points of the translation surface M ($m \geq 1$), S denotes the area of M, and s denotes the length of the shortest saddle connection of M.

For any $T > 0$ let $N_0(M, T)$ denote the number of saddle connection of M of length at most T. An effective upper estimate of this number was obtained in [Vo].

Theorem 5.1 ([Vo]) $N_0(M, T) \leq h_m s^{-2} T^2$ for any $T > 0$, where $h_1 = (3 \cdot 2^7)^6$ and $h_m = (400m)^{(2m)^{2m}}$ for $m \geq 2$.

In the case $m = 1$ the latter estimate can be significantly improved.

Lemma 5.2 Suppose M is a translation torus with a single (removable) singular point. Then $N_0(M, T) \leq 7s^{-2} T^2$ for any $T > 0$. In addition, $s^2 \leq 3S/2$.

Proof. The translation torus M is isometric to a torus $\mathbb{R}^2 / (v_1 \mathbb{Z} \oplus v_2 \mathbb{Z})$, where v_1 and v_2 are linearly independent vectors in $\mathbb{R}^2$ (we do not require that the isometry preserve directions). Let $\mathcal{L} = v_1 \mathbb{Z} \oplus v_2 \mathbb{Z}$. By $\mathcal{L}_0$ denote the set of vectors in $\mathcal{L}$ contained in neither of the lattices $2\mathcal{L}, 3\mathcal{L}, \ldots$. The isometry establishes a one-to-one correspondence between saddle connections of M and pairs of vectors $\pm v \in \mathcal{L}_0$. The length of a saddle connection is equal to the length of the corresponding vectors. By H denote the set of points $(y_1, y_2) \in \mathbb{R}^2$ such that either $y_2 > 0$, or $y_2 = 0$ and $y_1 > 0$. Then $N_0(M, T)$ is equal to the number of vectors of length at most T in the set $H \cap \mathcal{L}_0$. Notice that the vectors v_1 and v_2 are not determined in a unique way. Without loss of generality it can be assumed that $v_1 = (0, s)$ and $v_2 = (S/s, y)$, where $0 \leq y < s$. Then $\min(|v_2|, |v_2 - v_1|) \leq \sqrt{(S/s)^2 + (s/2)^2}$. Since $|v| \geq s$ for any nonzero vector $v \in v_1 \mathbb{Z} \oplus v_2 \mathbb{Z}$, we have $s^2 \leq (S/s)^2 + (s/2)^2$. It follows that $s^2 \leq 2S/\sqrt{3} \leq 3S/2$.

Let i and j be positive integers. The rectangle $P_{i,j}^+ = ((i-1)S/s, iS/s] \times [(j-1)s, js) \subset \mathbb{R}^2$ is contained in the halfplane H and contains precisely one element of the lattice $\mathcal{L}$. Likewise, the rectangle $P_{i,j}^- = [-iS/s, -(i-1)S/s) \times ((j-1)s, js]$ is contained in H and contains only one element of $\mathcal{L}$. Given $T > 0$, let $B_T = \{v \in \mathbb{R}^2 : |v| \leq T\}$. The number of rectangles of the form $P_{i,j}^\pm$ contained in the halfdisc $B_T \cap H$ does not exceed $\pi T^2/(2S)$. For any $k \in \mathbb{Z}$ set $L_k = \{(y_1, y_2) \in H : y_1 = Sk/s\}$. If $k \neq 0$, then the halfline L_k containes at most one element $v \in \mathcal{L}$ such that $v \in B_T$ but the rectangle of the form $P_{i,j}^\pm$ containing v is not contained in B_T. The halfline L_0 contains at most T/s elements of $B_T \cap \mathcal{L}$. Finally, the cardinality of the set $B_T \cap H \cap \mathcal{L}$ is at most $\pi T^2/(2S) + 2sT/S + T/s$. As this cardinality is not less than $N_0(M, T)$, we have $N_0(M, T) \leq \pi T^2/(2S) + 2sT/S + T/s \leq 3\pi s^{-2}T^2/4 + 4s^{-1}T \leq 3s^{-2}T^2 + 4s^{-1}T$. It follows that $N_0(M, T) \leq 7s^{-2}T^2$ for $T \geq s$. If $T < s$, then $N_0(M, T) = 0 < 7s^{-2}T^2$. ∎

The following lemma is an improved version of Theorem 1.3 for translation tori.

Lemma 5.3 *Suppose M is a translation torus with $m \geq 1$ singular points. Then there exists a cylinder of periodic geodesics of length at most $2\sqrt{S}$ and of area at least S/m.*

Proof. Let ω denote the translation structure of the translation torus M. Let $x_1, x_2, \ldots, x_m$ be the singular points of ω. All singular points are removable. By ω_1 denote the translation structure on M such that $\omega_1 \supset \omega$ and x_1 is the only singular point of ω_1. Let M_1 denote the torus M considered as the translation surface with the translation structure ω_1. Let S_1 be the area of M_1 and s_1 be the length of the shortest saddle connection of M_1. It is easy to observe that $S_1 = S$ and $s_1 \geq s$. The shortest saddle connection of M_1 bounds a periodic cylinder Λ of M_1. The length of Λ is equal to s_1 and the area of Λ is equal to S. By Lemma 5.2, $s_1 \leq \sqrt{3S/2} < 2\sqrt{S}$. The points $x_2, \ldots, x_m$ split the cylinder Λ into several periodic cylinders of the translation surface M. All these cylinders are of length $s_1 \leq 2\sqrt{S}$. The number of the cylinders does not exceed m, hence at least one of them is of area not less than S/m. ∎

Proof of Theorem 1.8. Let M be a translation surface. To each cylinder of periodic geodesics of M we assign a saddle connection bounding the cylinder. The length of the saddle connection does not exceed the length of the cylinder. It is possible that a saddle connection bounds two different periodic cylinders. Nevertheless the assignment can be done so that any saddle connection is assigned to at most one cylinder. It follows that $N_1(M, T) \leq N_0(M, T)$ for any $T > 0$. Thus Theorem 5.1 (in the case $m \geq 2$) and Lemma 5.2 (in the case $m = 1$) imply that $N_1(M, T) \leq (400m)^{(2m)^{2m}} s^{-2}T^2$ for any $T > 0$. Besides, the estimate $N_2(M, T)/S \leq N_1(M, T)$ is trivial.

We proceed to the proof of the lower estimate of $N_2(M, T)$. Let $C_0 = \tilde{h}_m s^{-2}$, where $\tilde{h}_1 = 7$ and $\tilde{h}_m = (400m)^{(2m)^{2m}}$ for $m \geq 2$. By Theorem 5.1 and Lemma 5.2, $N_1(M, T) \leq N_0(M, T) \leq C_0 T^2$ for any $T > 0$. Denote by $\sigma(T)$ the sum of inverse lengths over all cylinders of periodic geodesics of length at most T. Let $T_1 \leq T_2 \leq \ldots \leq T_n \leq \ldots$ be lengths of periodic cylinders of M in ascending order. It follows from the estimate $N_1(M, T) \leq C_0 T^2$ that $T_n \geq C_0^{-1/2} n^{1/2}$, $n = 1, 2, \ldots$. Therefore,

$$\sigma(T) = \sum_{n : T_n \leq T} T_n^{-1} \leq C_0^{1/2} \sum_{n : T_n \leq T} n^{-1/2} \leq C_0^{1/2} \sum_{n \leq C_0 T^2} n^{-1/2} \leq$$

$$\leq C_0^{1/2} \int_0^{C_0 T^2} x^{-1/2} \, dx = C_0^{1/2} \cdot 2(C_0 T^2)^{1/2} = 2 C_0 T.$$

Set $T_0 = l_m \sqrt{S}$, where $l_1 = l_2 = 2$ and $l_m = 2^{2^{4m}}$ for $m \geq 3$. Let Λ be a periodic cylinder and $|\Lambda|$ be the length of Λ. For any $\lambda \geq 1$ let $A_\Lambda(\lambda)$ denote the set of directions $v \in S^1$ such that the projection of periodic geodesics from Λ on the direction orthogonal to v is at most $\lambda^{-1} T_0$. Let $v \in A_\Lambda(\lambda)$ and φ be the angle between v and the direction of the cylinder Λ ($0 \leq \varphi \leq \pi/2$). Then $\varphi \leq \pi/2 \cdot \sin \varphi \leq \pi/2 \cdot \lambda^{-1} T_0 / |\Lambda|$. It follows that

$$\nu(A_\Lambda(\lambda)) \leq 4 \cdot \pi/2 \cdot \lambda^{-1} T_0 / |\Lambda| = 2\pi T_0 \lambda^{-1} \cdot |\Lambda|^{-1},$$

where ν is Lebesgue measure on the circle S^1 normalized so that $\nu(S^1) = 2\pi$.

Take an arbitrary number $T \geq T_0$ and set $\lambda = T/T_0$. Let ω denote the translation structure of the translation surface M. For any direction $v \in S^1$ define an operator $a_{\lambda,v} \in \mathrm{SL}(2, \mathbb{R})$ by equalities $a_{\lambda,v} v = \lambda^{-1} v$, $a_{\lambda,v} u = \lambda u$, where $u \in S^1$ is a vector orthogonal to v. We claim that there exists a periodic cylinder Λ of area at least S/m such that the length of Λ with respect to the translation structure $a_{\lambda,v} \omega$ does not exceed T_0. In the case $m \geq 3$, this follows from Theorem 1.3. In the case $m \leq 2$, the translation surface M is a torus, thus the claim follows from Lemma 5.3. The projection of a geodesic from Λ on the direction u (with respect to the translation structure ω) is at most $\lambda^{-1} T_0$, hence $v \in A_\Lambda(\lambda)$. Since $\lambda \geq 1$, the condition number (see Section 3) of the operator $a_{\lambda,v}$ is equal to λ. Therefore, $|\Lambda| \leq \lambda T_0 = T$. It follows that the union of sets $A_\Lambda(\lambda)$ over all periodic cylinders Λ of length at most T (with respect to ω) and of area at least S/m is the circle S^1.

Let Σ denote the sum of measures of sets $A_\Lambda(\lambda)$, where Λ runs over all periodic cylinders of length at most T and of area at least S/m. Since these sets cover the whole circle S^1, we have $\Sigma \geq 2\pi$. Set $\alpha = (4T_0^2 C_0)^{-1}$. By Theorem 1.3 and Lemma 5.3, the translation structure ω admits a periodic cylinder of length at most T_0, hence $1 \leq N_1(M, T_0) \leq C_0 T_0^2$. In particular, $\alpha \leq 1/4$. The sum Σ can be written as $\Sigma_1 + \Sigma_2$, where Σ_1 is the sum of summands corresponding to the cylinders of length at most αT, and Σ_2 is

the sum over cylinders of length greater than αT. It follows from the above estimate of $\nu(A_\Lambda(\lambda))$ that

$$\Sigma_1 \leq 2\pi T_0 \lambda^{-1} \cdot \sigma(\alpha T) \leq 2\pi T_0 \lambda^{-1} \cdot 2C_0 \alpha T,$$

$$\Sigma_2 \leq 2\pi T_0 \lambda^{-1} \cdot (\alpha T)^{-1} N(T),$$

where $N(T)$ is the number of periodic cylinders of length at most T and of area at least S/m. Then

$$2\pi \leq 2\pi T_0 \lambda^{-1}(2C_0 \alpha T + (\alpha T)^{-1} N(T)),$$

thus,

$$N(T) \geq (T_0^{-1}\lambda - 2C_0 \alpha T)(\alpha T) = (8T_0^4 C_0)^{-1} T^2.$$

Consequently,

$$N_2(M,T)/S \geq N(T)/m \geq (8mT_0^4 C_0)^{-1} T^2 = (8ml_m^4 \tilde{h}_m)^{-1} s^2 S^{-2} T^2.$$

To complete the proof, it remains to show that $8ml_m^4 \tilde{h}_m \leq (600m)^{(2m)^{2m}}$. If $m = 1$, then $8ml_m^4 \tilde{h}_m = 7 \cdot 2^7 < 600^4$. If $m = 2$, then $8ml_m^4 \tilde{h}_m = 2^8 \cdot 800^{4^4} < 1200^{4^4}$. In the case $m \geq 3$, we have $(m/2)^{2m} \geq (3/2)^6 > 10$. Then $(2m)^{2m} > 10 \cdot 4^{2m} = 10 \cdot 2^{4m}$. It follows that $(3/2)^{(2m)^{2m}} > 2^{5 \cdot 2^{4m}}$. Besides, $2^{4m} > 8m$. Finally,

$$(8ml_m^4 \tilde{h}_m)^{-1}(600m)^{(2m)^{2m}} = (8m)^{-1} \cdot 2^{-4 \cdot 2^{4m}} \cdot (3/2)^{(2m)^{2m}} >$$

$$(8m)^{-1} \cdot 2^{2^{4m}} > 2^{2^{4m}} \cdot 2^{-4m} > 2^{4m} > 1.$$

The theorem is proved. ∎

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