**6 Rapid #: -1840065**

**Ariel**
**IP: 128.194.84.50**

| Status | Rapid Code | Branch Name | Start Date |
|---|---|---|---|
| Pending | NED | Snell Library | 6/2/2008 2:22:38 PM |

**CALL #:** **QA1 .M3400**
**LOCATION:** **NED :: Snell Library :: sp**
TYPE: Article CC:CCL
JOURNAL TITLE: Mathematical notes of the Academy of Sciences of the USSR
USER JOURNAL TITLE: Mathematical Notes
NED CATALOG TITLE: Mathematical notes = Matematicheskie zametki.
ARTICLE TITLE: On the measure of the set of periodic points of the billiard
ARTICLE AUTHOR: Ya.B.Vorobets
VOLUME: 55
ISSUE: 5
MONTH:
YEAR: 1994
PAGES: 455-460
ISSN: 0001-4346
OCLC #:
CROSS REFERENCE ID: 1315304
VERIFIED:

**BORROWER:** **TXA :: Main Library**
**PATRON:** **Vorobets,Yaroslav**
PATRON ID:
PATRON ADDRESS:
PATRON PHONE:
PATRON FAX:
PATRON E-MAIL:
PATRON DEPT: College of Science
PATRON STATUS: Faculty
PATRON NOTES:

Mathematical Notes, Vol. 55, No. 5, 1994

# ON THE MEASURE OF THE SET OF PERIODIC POINTS OF THE BILLIARD

Ya. B. Vorobets

Let G be a bounded domain in Euclidean space $\mathbb{R}^n$ ($n \geq 2$) with boundary consisting of a finite number of piecewise $C^1$-smooth surfaces. We define a billiard cascade for G as follows. The phase space M of the cascade is formed by linear elements $x = (q, v) \in \partial \times S^{n-1}$, where q is a nonsingular point of the boundary $\partial G$ (the so-called support of the element x), and v is a unit vector at q directed toward the interior of G, i.e., $\langle n(q), v\rangle > 0$ (here n(q) is the interior unit normal vector at q on $\partial G$, and $\langle \cdot,\cdot \rangle$ is the scalar product in $\mathbb{R}^n$). M is an open set in $\partial G \times S^{n-1}$, and its boundary $\partial M$ is a compact set of codimension 1. M inherits the natural measure $d\mu(x) = d\rho(q)d\omega(v)$ from $\partial G \times S^{n-1}$, where $d\rho$ is a volume element of $\partial G$, and $\omega$ is the Lebesgue measure on $S^{n-1}$.

For an arbitrary element $x = (q, v) \in M$, we draw, from the point q, a ray in the direction of the vector v, and we let p be the first point (other than q) on the boundary $\partial G$ through which this ray passes (we assume that for each $x \in M$, such a ray exists). The transformation T of the space M is defined on the element x if p is a nonsingular boundary element and the ray we have drawn is not tangent to the boundary at p. Here we assume that $Tx = (p, u)$, where u is the vector obtained from v by reflection at the hyperplane tangent to $\partial G$ at the point p, i.e., $u = v - 2\langle\langle v, n(p)\rangle n(p)$. It is clear that $Tx \in M$.

Let $M_1$ be the region defined by the transformation T, $M_{-1} = T(M_1)$. It is clear that $M_1$ and $M_{-1}$ are open subsets of M, and T homeomorphically maps $M_1$ onto $M_{-1}$. In general, for an arbitrary $m \in \mathbb{Z}$, we use $M_m$ to denote the set of those elements of M on which the transformation $T^m$ is defined. Each $M_m$ is an open subset of M with complement in $\bar{M}$ a compact set of codimension 1. The set $M_\infty = \bigcap_{m\in\mathbb{Z}} M_m$, on which the entire cascade $\{T^m\}$ is a subset of complete measure in M. T bijectively maps $M_\infty$ onto itself.

For an arbitrary element $z = (q_0, v_0) \in M$, consider the two-sided sequence $\dots, T^{-m}x = (q_{-m}, v_{-m}), \dots, T^{-1}x = (q_{-1}, v_{-1}), \dots, x = (q_0, v_0), \dots, T_x = (q_1, v_1), \dots, T^m x = (q_m, v_m), \dots$ (if $x \notin M_\infty$, this sequence will not be infinite on both sides), which we call the trajectory of the element x, or the billiard trajectory leaving the point $q_0$ in the direction $v_0$. The points $q_m$ are called the vertices of the trajectory, and the segments $[q_m, q_{m+1}]$ are its arcs. The sequence of vertices $\dots, q_{-m}, \dots, q_{-1}, q_0, q_1, \dots, q_m, \dots$ completely determines a trajectory, so we will also call it a billiard trajectory.

**Definition.** An element $x \in M$ is said to be a periodic billiard point in G if $T^m x = x$ for some $m > 0$. We call the smallest such m the period of the point x. In this case the trajectory of the point x is called a periodic billiard trajectory.

We have the following known

**Conjecture.** A billiard in an arbitrary region G is aperiodic — the set of its periodic points has measure zero in the phase space.

We should note that there are no periodic points of period 1 (stationary elements) under the transformation T (they appear upon compactification of the phase space, forming, in this case, a set of codimension 1, i.e., a set of measure zero). Further, in a periodic trajectory of period 2, both arcs coincide, so they are perpendicular to the boundary of the region at their vertices (here the fact the space is Euclidean is essential, since for a spherical billiard our argument is false). Thus, an arbitrary point on the boundary $\partial G$ is the vertex of no more than one periodic trajectory of period 2, so the elements of period 2 in M are contained in a set of positive codimension and measure zero.

The first nontrivial case is that of periodic points of period 3.

**THEOREM.** The periodic points of a billiard with period 3 form a set of measure zero.

For a billiard in a plane region with piecewise $C^3$-smooth boundary, our proposition was proved in [1, 2]. In the present paper, we will give a complete proof of the theorem. It is based on considerations different from those used in [1, 2].

M. V. Lomonosov Moscow State University. Translated from Matematicheskie Zametki, Vol. 55, No. 5, pp. 25-35, May, 1994. Original article submitted June 15, 1993.

0001-4346/94/5556-0455$12.50  

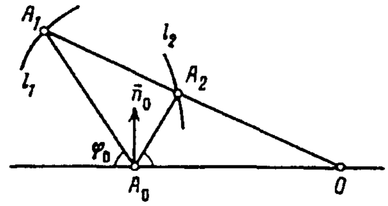


Fig. 1

**Section 1.** Let $A_0$, $A_1$, and $A_2$ be 3 noncolinear points in a plane, and let $l_0$ be the line passing through the point $A_0$ that is orthogonal to the vector $\bar{n}_0 = \frac{\overline{A_0A_1}}{|A_0A_1|} + \frac{\overline{A_0A_2}}{|A_0A_2|}$. Let the lines $A_1A_2$ and $l_0$ intersect at some point O. Through those points of $A_1$ and $A_2$ that are located close to the point O, we pass a hyperbola with foci at the points $A_0$ and O, and through another point, we pass an ellipse with the same foci. If the lines $A_1A_2$ and $l_0$ are parallel, we draw an arc of a parabola with focus at the point $A_0$ and axis $l_0$ through the points $A_1$ and $A_2$. We denote the curves passing through the points $A_1$ and $A_2$, respectively, by $l_1$ and $l_2$ (Fig. 1).

**LEMMA 1.** Let G be a bounded plane region with piecewise $C^1$-smooth boundary. Assume that the points $A_0$, $A_1$, and $A_2$ belong to the boundary of the region G, and the tangent to the point $A_0$ in $\partial G$ is $l_0$, while in the neighborhoods of $A_1$ and $A_2$, the boundary $\partial G$ coincides with the curves $l_1$ and $l_2$, respectively. Then the points $A_0$, $A_1$, and $A_2$ form a periodic billiard trajectory of period 3 for G, and so are all trajectories leaving the point $A_0$ in close directions.

**Proof.** We begin with the case in which the lines $A_1A_2$ and $l_0$ intersect. By construction, the lines $A_0A_1$ and $A_0A_2$ form equal angles $\varphi_0$ with the line $l_0$, where $0 < \varphi_0 < \pi/2$. For an arbitrary $\varphi$ sufficiently close to $\varphi_0$, there exist points $B_1 \in l_1$ and $B_2 \in l_2$ such that the segments $A_0B_1$ and $A_0B_2$ meet the line $l_0$ in the angle $\varphi$. Suppose, for definiteness, the point $A_2$ belongs to the segment $A_1O$. Then, by the construction of the curves $l_1$ and $l_2$, we have $|A_0B_1| + |OB_1| = D$, $|OB_2| - |A_0B_2| = E$, where $D = |A_0A_1| + |OA_1|$, $E = |OA_2| - |A_0A_2|$. It follows from our construction that $E > 0$, and it follows from the triangle inequality that $D > C > E$, where $C = |A_0O|$.

We set $r_1 = |A_0B_1|$, $r_2 = |A_0B_2|$. Then

$$|OB_1| = \sqrt{(C + r_1\cos\varphi)^2 + (r_1\sin\varphi)^2},$$
$$|OB_2| = \sqrt{(C - r_2\cos\varphi)^2 + (r_2\sin\varphi)^2}.$$

Substituting these expressions into the equations that we have, we obtain, after elementary transformations,

$$r_1 = \frac{D^2 - C^2}{2(D + C\cos\varphi)}, \qquad r_2 = \frac{C^2 - E^2}{2(E + C\cos\varphi)}.$$

The three points O, $B_1$, and $B_2$ are collinear if and only if $\frac{C + r_1\cos\varphi}{r_1\sin\varphi} = \frac{C - r_2\cos\varphi}{r_2\sin\varphi}$. For $\varphi$ close to $\varphi_0$ we have $\sin\varphi \neq 0$, and this conditions is equivalent to the condition $C(1/r_2 - 1/r_1) = 2\cos\varphi$. Substitution for $r_1$ and $r_2$ and elementary operations leads to the following equation:

$$\frac{C/E - D/C}{(D/C + 1)(C/E + 1)} \cdot \left(1 + (1 + \cos\varphi) \cdot \frac{C/E + D/C}{(D/C - 1)(C/E - 1)}\right) = 0.$$

This equation holds when $\varphi = \varphi_0$, since, in this case, $B_1 = A_1$ and $B_2 = A_2$. However, the inequality $D > C > E > 0$ implies that $1 + (1 + \cos\varphi) \cdot \frac{C/E + D/C}{(D/C - 1)(C/E - 1)} > 1$, so $C/E = D/C$, and the equation we have given is identically true. Thus, for any $\varphi$ close to $\varphi_0$, the points $B_1$, $B_2$, and 0 are collinear. In view of the focal properties of ellipses and hyperbolas, it follows that $A_0$, $B_1$, and $B_2$ constitute a billiard trajectory of period 3.

When the lines $A_1A_2$ and $l_0$ are parallel, we choose, as before, points $B_1 \in l_1$ and $B_2 \in l_2$ so that the segments $A_0B_1$ and $A_0B_2$ intersect the line $l_0$ in equal angles $\varphi$. Since the lines $l_1$ and $l_2$ are symmetric with respect to a line passing through $A_0$ perpendicular to the point $l_0$, the points $B_1$ and $B_2$ are also symmetric with respect to it. Now, the line $B_1B_2$ is parallel to $l_0$, and it follows from the focal properties of parabolas that $A_0$, $B_1$, and $B_2$ constitute a billiard trajectory of period 3. The lemma is thus proved.

In the case of dimensions $n \geq 3$, we conduct all of the above constructions in the two-dimensional plane $\pi$ passing through the points $A_0$, $A_1$, and $A_2$. Then we apply, to $\pi$, all possible rotations of the space $\mathbb{R}^n$ about the line passing through the point $A_0$ and the vector $\bar{n}_0$. In this case, the line $l_0$ and the curves $l_1$ and $l_2$ respectively determine in space a hyperplane $L_0$ orthogonal to the vector $\bar{n}_0$ and certain 4-th order surfaces $L_1$ and $L_2$.

**LEMMA 1′.** Lemma 1 remains true when $L_0$, $L_1$, and $L_2$ are substituted for $l_0$, $l_1$, and $l_2$.

**Proof.** In the cross-sections of the surfaces $L_0$, $L_1$, and $L_2$ generated by a 2-dimensional plane passing through the point $A_0$ and the vector $\bar{n}_0$ we obtain the configuration considered in Lemma 1, i.e., all of the billiard trajectories beginning at the point $A_0$ are periodic with period 3. However, by construction, each such cross-section is normal (with each point on the surface, it contains the normal to it at this point), so a periodic billiard trajectory of period 3 in the cross-section is also a periodic billiard trajectory of period 3 in the entire space.

**Section 2. Definition.** Let E be a measurable subset in $\mathbb{R}^m$. A point $x \in E$ is called a point of density of the set E if

$$\frac{\lambda(B(x,\varepsilon)\cap E)}{\lambda(B(x,\varepsilon))} \to 1 \quad \text{as} \quad \varepsilon \to +0.$$

(here $B(x, \varepsilon)$ is the ball of radius $\varepsilon$ with center at the point x and $\lambda$ is the Lebesgue measure in $\mathbb{R}^m$). Now, let E be a measurable subset of a continuously differentiable manifold X. A point $x \in E$ is said to be a point of density of E if it is such in some (and therefore any) smooth local coordinate system in the neighborhood of the point x. It is known that almost all points of a measurable subset $E \subset X$ are points of density [3, p. 209]. Further, let X be the direct product of two manifolds Y and Z. For an arbitrary $y \in Y$, consider the set $E_y = \{z \in Z \mid (y, z) \in E\}$, the Y-cross-section of the set E. We say that a point $(y, z) \in E$ is a point of density of the set E in the Y-cross-section if $E_y$ is a measurable subset of Z that has z as a point of density.

**LEMMA 2.** Almost all points of a measurable set $E \subset Y \times Z$ are points of density in the Y-cross-section.

**Proof.** Without loss of generality, we can assume that $Y = \mathbb{R}^l$, $Z = \mathbb{R}^m$. Let F be the set formed by the points of density of the set E in the Y-cross-section. For almost all $y \in Y$, the set $E_y$ is measurable in Z, so the sets $E_y$ and $F_y$ differ by a set of measure zero, since $F_y$ consists of the points density of $E_y$. Thus, in view of the Fubini theorem, the lemma will be proved if we demonstrate that F is measurable.

For an arbitrary $\varepsilon > 0$, consider the function $\chi_\varepsilon$ defined on $Y \times Z$ by the expression $\chi_\varepsilon(y,z) = \dfrac{\lambda(B(z,\varepsilon)\cap E_y)}{\lambda(B(z,\varepsilon))}$,

where $B(z,\varepsilon)$ is a ball in $\mathbb{R}^m$, and $\lambda$ is the Lebesgue measure in $\mathbb{R}^m$. The function $\chi_\varepsilon$ is defined everywhere on $Y \times Z$ outside some set of measure 0 that is independent of the choice of $\varepsilon$. We will show that for any measurable set E, this function is measurable. Indeed, if E is the monotonic limit of a sequence of sets obtained in the indicated manner, then $\chi_\varepsilon$ is almost everywhere the limit of a sequence of measurable functions, so it is itself measurable. Furthermore, the measurability of the function $\chi_\varepsilon$ is not affected if we remove some subset of measure zero from E. Finally, the indicated operations make it possible to obtain an arbitrary measurable set from elementary sets — unions of a finite number of parallelepipeds — for which the measurability of the function $\chi_\varepsilon$ is obvious.

A point $x \in E$ belongs to the set F if $\chi_\varepsilon(x) \to 1$ as $\varepsilon \to +0$. When $1/(k+1) \leq \varepsilon \leq 1/k$, $k \in \mathbb{N}$, we have

$$\chi_{1/(k+1)}(x)\cdot\lambda_k^{-1} \leq \chi_\varepsilon(x) \leq \chi_{1/k}(x)\cdot\lambda_k, \text{ where } \lambda_k = \left(\frac{k+1}{k}\right)^m.$$

Since $\lambda_k \to 1$ as $k \to \infty$, the condition $\chi_\varepsilon(x) \to 1$ as $\varepsilon \to +0$ is formally equivalent to the weaker condition $\lim_{k\to\infty} \chi_{1/k}(x) = 1$. Thus, F is a set of point in E in which a sequence of measurable functions converges to 1, and so is itself measurable. The lemma is proved.

**Section 3. Definition.** Let E be a measurable subset of $\mathbb{R}^m$, and let f be a vector function defined on E. We say that the function f is differentiable with respect to E at the point $x \in E$ if it is the restriction to E of some function g that is differentiable at the point x. By the differential of a function g at the point x we mean the differential of a function f at the point x with respect to the set E. If x is a point of density of the set E, this differential is uniquely defined. We say that the function f is differentiable on the set E if it is differentiable with respect to E at any point in E. In this case we can define the second differential of the function f with respect to E, etc. In particular, we say that a function f is k-times differentiable ( $| k \geq 1 |$ ) on the set E if there exist differentiable (on E) operator-valued functions $f_0 = f, f_1, \ldots, f_{k-1}$ such that $df_i(x) = f_{i+1}(x)$, $0 \leq$

$i \leq k-2$ for all $x \in E$. By the differential $df_{k-1}(x)$ we mean the k-th differential of the function f at the point x with respect to the set E. $d^kf(x)$ is uniquely defined, generally speaking, only at points of density of the set E. It is clear that a function f that is k-times differentiable in the usual sense in the neighborhood of the set E is also k-times differentiable on the set E, and its differentials of up to order k with respect to the set E are the same as the usual differentials at the points of this set.

If two functions f and g are defined on the set E, take values in $\mathbb{R}^l$, and are k-times differentiable on E, the linear combination $\alpha F + \beta g$ (and in the case $l = 1$ the product fg) are also k-times differentiable on E, while their differentials with respect to the set E are defined (at points of density of the set E, uniquely) by the values of the functions f, g, and their differentials. Furthermore, if a function f with range in $\mathbb{R}^l$ is k-times differentiable on a measurable set $E \subset \mathbb{R}^m$, and the function g is k-times differentiable on a set $F \subset \mathbb{R}^l$, where $f(E) \subset F$, then the composition $g \circ f$ is k-times differentiable on the set E, while its differential at an arbitrary point $x \in E$ is determined by the differentials of the functions f and g at the points x and f(x), respectively. The proofs of these claims are the same as the proofs of the analogous propositions of ordinary differential calculus.

**LEMMA 3.** Let g be a function with range in $\mathbb{R}^l$ that is differentiable on a measurable set $E \subset \mathbb{R}^m$, and let H be a function that is infinitely differentiable on a measurable set $F \subset \mathbb{R}^m \times \mathbb{R}^l$. Suppose, further, that for any $x \in E$, the point (x, g(x)) belongs to the set F, and $dg(x) = H(x, g(x))$. Then the function g is infinitely differentiable at the point $x \in E$, and its differentials at the point $x \in E$ are defined (uniquely, if x is a point of density for E) by the value $y = g(x)$, as well as the values of the function H and its differentials with respect to the set F at the point (x, y).

**Proof.** We set $H_1 = H$, and, by induction for $k \geq 1$, $H_{k+1} = \frac{\partial H_k}{\partial x} + \frac{\partial H_k}{\partial y} H$. Then $H_1, H_2, \ldots$, are infinitely differentiable operator-valued functions on F. We proceed by induction on $k \in N$ to show, using the theorem on the differential of a composite, that the function g is k-times differentiable on the set E, and its k-th differential at the point $x \in E$ with respect to E is $d^kg(x) = H_k(x, g(x))$. To complete the proof, we need only add that, by the construction of the functions $H_1, H_2, \ldots$, their values at the point (x, g(x)) are defined by the values of the function H and its differentials at this point.

**Section 4.** Let G be a bounded domain of $\mathbb{R}^n$ with piecewise $C^1$-smooth boundary, Let M be the phase space of a billiard cascade for G, and let S be the set of elements of M that are periodic points of a billiard of period 3. The set S is closed in M. Let $x = (A_0, v)$ be an element of S, and let $A_0$, $A_1$, and $A_2$ be the vertices of the trajectory. Through the points $A_1$ and $A_2$ we pass surfaces $L_1$ and $L_2$ as in Section 1 (in the plane case, the curves $l_1$ and $l_2$ are taken to be $L_1$ and $L_2$).

Let $(\xi; \xi^n) = (\xi^1, \ldots, \xi^{n-1}; \xi^n)$ be a $C^\infty$-smooth coordinate system in the neighborhood of the point $A_1$. We use $(\xi_0, \xi_0{}^n)$ to denote the coordinates of the point $A_1$ in this system. Also, assume that the $\xi^n$ axis is transverse to the boundary $\partial D$ at the point $A_1$. Then, in the neighborhood of the point $A_1$, the boundary $\partial G$ and the surface $L_1$ are specified by the equations $\xi^n = f(\xi)$ and $\xi^n = \tilde{f}(\xi)$, where f is a $C^1$-smooth function and $\tilde{f}$ is a $C^\infty$-smooth function in the neighborhood of the point $\xi_0 \in \mathbb{R}^{n-1}$ and $\xi_0{}^n = f(\xi_0) = \tilde{f}(\xi_0)$.

**LEMMA 4.** If the element x is a point of density of the set S in the $\partial G$-section, then the function f is infinitely differentiable on some measurable set E for which $\xi_0$ is a point of density, and all differentials of the function with respect to the set E at the point $\xi_0$ coincide with the corresponding (ordinary) differentials of the function $\tilde{f}$ at the point $\xi_0$.

In addition, the analogous proposition holds for the point $A_2$ in the surface $L_2$.

**Proof.** We assume that the lemma has been proved for the system $(\xi; \xi^n)$, and we will prove it for an arbitrary coordinate system $(\eta; \eta^n) = (\eta^1, \ldots, \eta^{n-1}; \eta^n)$ that satisfies the requirements imposed on the system $(\xi; \xi^n)$. Suppose that $(\eta_0; \eta_0{}^n)$ are the coordinates of the points $A_1$ in the new system, and $(h; h^n)$ is a function for transforming the coordinates $(\xi; \xi^n)$ into the coordinates $(\eta; \eta^n)$, i.e., $\xi = h(\eta; \eta^n)$, $\xi^n = h^n(\eta; \eta^n)$. The mapping $(h; h^n)$ is a $C^\infty$-smooth diffeomorphism of a neighborhood of the point $(\eta_0; \eta_0{}^n)$ of $\mathbb{R}^n$ into a neighborhood of the point $(\xi_0; \xi_0{}^n)$. The boundary $\partial G$ and the surface $L_1$ close to the point $A_1$ are given by the equations $\eta^n = g(\eta)$ and $\eta^n = \tilde{g}(\eta)$, where g is $C^1$-smooth and $\tilde{g}$ is $C^\infty$-smooth in the neighborhood of the point $\eta_0 \in \mathbb{R}^{n-1}$, $g(\eta_0) = \tilde{g}(\eta_0) = \eta^n_0$. We set $D(\eta; \eta^n) = h^n(\eta; \eta^n) - f(h(\eta; \eta^n))$. The function D is continuously differentiable in the neighborhood of the point $(\eta_0; \eta_0{}^n)$ and $dD(\eta_0; \eta_0{}^n) \neq 0$. In view of the identity $D(\eta; g(\eta)) = 0$, it follows that $\partial D/\partial\eta^n(\eta_0; \eta_0{}^n) \neq 0$, and

$$dg(\eta) = -\left(\frac{\partial D}{\partial \eta^n}(\eta; g(\eta))\right)^{-1} \cdot \frac{\partial D}{\partial \eta}(\eta; g(\eta))$$

in the neighborhood of the point $\eta_0$. The equation we have obtained can be transformed into the form $dg(\eta) = H(\eta, g(\eta), df(h(\eta; g(\eta))))$, where H and h are infinitely smooth vector functions. A similar argument leads to the equation $d\tilde{g}(\eta) = H(\eta, \tilde{g}(\eta), d\tilde{f}(h(\eta; \tilde{g}(\eta))))$.

Let F be the preimage of the set E under the mapping $h_1(\eta) = h(\eta; g(\eta))$. In the neighborhood of the point $\eta_0$, this mapping is a diffeomorphism, so F is a measurable set and $\eta_0 = h_1^{-1}(\xi_0)$ is a point of density for it. Using Lemma 3 and the remark preceding it, we can use the two equations we have derived and the relations $g(\eta_0) = \tilde{g}(\eta_0)$, $h_1(\eta_0) = \xi_0$, to prove infinite differentiability for the function g on the set F, and equality of the differentials of the function g at the point $\eta_0$ with respect to F to the differentials of the function $\tilde{g}$ at $\eta_0$.

We will now prove the lemma in a specially selected coordinate system. Let $x^1, \ldots, x^n$ be Cartesian coordinates in $\mathbb{R}^n$ with origin at the point $A_0$ and the property that the vector $\bar{n}_0$ (see Sec. 1) points in the positive direction of the $x^n$ axis. The coordinates in which we are interested, $(y, r) = (y^1, \ldots, y^{n-1}, r)$, are related to the Cartesian coordinates in the following way: $x^1 = ry^1, \ldots, x^{n-1} = ry^{n-1}$, $x^n = r$. In this coordinate system the points $A_1$ and $A_2$ have the coordinates $(y_0, r_{10})$ and $(-y_0, r_{20})$, respectively, where $y_0 \in \mathbb{R}^{n-1}$, and $r_{10}$ and $r_{20}$ are positive numbers. The boundary $\partial G$ in the neighborhood of the points $A_1$ and $A_2$ is given, in the (y, r) coordinates, by the equations $r = r_1(y)$ and $r = r_2(-y)$, while the surfaces $L_1$ and $L_2$ are given by the equations $r = \tilde{r}_1(y)$ and $r = \tilde{r}_2(-y)$, where the functions $r_1$ and $r_2$ are singly differentiable in some neighborhood U of the point $y_0$, and the functions $\tilde{r}_1$ and $\tilde{r}_2$ are infinitely continuously differentiable in the neighborhood U of the point $y_0$.

For an arbitrary $y \in U$, we use $B_1$ and $B_2$ to denote the points of the boundary $\partial G$ with coordinates $(y, r_1(y))$ and $(-y, r_2(y))$, respectively. Then the points $B_2$, $A_0$, and $B_1$ form segments of a billiard trajectory. We set

$$\bar{n}_1 = \frac{\overline{A_0B_1}}{|A_0B_1|} + \frac{\overline{B_2B_1}}{|B_2B_1|}, \qquad \bar{n}_2 = \frac{\overline{A_0B_2}}{|A_0B_2|} + \frac{\overline{B_1B_2}}{|B_1B_2|}.$$

Treating $\overline{A_0B_1}, \overline{A_0B_2}, \bar{n}_1$, and $\bar{n}_2$ as vector functions of y, we also set

$$f_{1i}(y) = \left\langle \frac{\partial}{\partial y^i}\overline{A_0B_1}, \bar{n}_1 \right\rangle, f_{2i}(y) = \left\langle \frac{\partial}{\partial y^i}\overline{A_0B_2}, \bar{n}_2 \right\rangle,$$

$1 \le i \le n$, where $\langle \cdot,\cdot \rangle$ is the scalar product in $\mathbb{R}^n$. Writing all vectors in the coordinate system $x^1, \ldots, x^n$ and using the fact that this system is Cartesian, we find that

$$f_{1i}(y) = D \cdot r_1 y^i + D_1 \cdot \frac{\partial r_1}{\partial y^i}, \quad f_{2i}(y) = D \cdot r_2 y^i + D_2 \cdot \frac{\partial r_2}{\partial y^i},$$

$1 \le i < n$, where

$$D = \frac{1}{\sqrt{\|y\|^2+1}} + \frac{r_1 + r_2}{\sqrt{\|y\|^2(r_1+r_2)^2 + (r_1-r_2)^2}},$$
$$D_1 = \sqrt{\|y\|^2+1} + \frac{\|y\|^2(r_1+r_2) + (r_1-r_2)}{\sqrt{\|y\|^2(r_1+r_2)^2 + (r_1-r_2)^2}},$$
$$D_2 = \sqrt{\|y\|^2+1} + \frac{\|y\|^2(r_1+r_2) + (r_2-r_1)}{\sqrt{\|y\|^2(r_1+r_2)^2 + (r_1-r_2)^2}}.$$

Let E be the set of those $y \in U$ for which the points $B_2$, $A_0$, and $B_1$ form a periodic billiard trajectory of period 3. E is a measurable set (it is closed in U). It follows from the conditions of the lemma that $y_0$ is a point of density of the set E. It is clear that $y \in E$ if and only if the vectors $\bar{n}_1(y)$ and $\bar{n}_2(y)$ are normals to the boundary $\partial G$ at the points $B_1$ and $B_2$, respectively, or, equivalently, when $f_{1i}(y) = f_{2i}(y) = 0$, $1 \le i < n$. We should note that the equations $D_1 = 0$ and $D_2 = 0$ are equivalent to the conditions $|A_0B_2| = |A_0B_1| + |B_1B_2|$ and $|A_0B_1| = |A_0B_2| + |B_2B_1|$, respectively. When $y = y_0$, these last do not hold, since the points $A_0$, $A_1$, and $A_2$ are not collinear, so in the neighborhood of the point $y_0$, the system of equations $f_{1i}(y) = 0$, $f_{2i}(y) = 0$, $1 \le i < n$, can be reduced to the form

$$dr_1(y) = H_1(y, r_1(y), r_2(y)), \quad dr_2(y) = H_2(y, r_1(y), r_2(y)),$$

where $H_1$ and $H_2$ are real-analytic vector functions in the neighborhood of the points $(y_0, r_{10}, r_{20}) \in \mathbb{R}^{n-1} \times \mathbb{R} \times \mathbb{R}$. As we have already noted, the equations we have written hold when $y \in E$. The equations obtained from then by substituting $\tilde{r}_1$ for

$r_1$ and $\tilde{r}_2$ for $r_2$ are identically true in the neighborhood of the point $y_0$, which follows from Lemmas 1 and 1′. Noting that $r_1(y_0) = \tilde{r}_1(y_0) = r_{10}$, $r_2(y_0) = \tilde{r}_2(y_0) = r_{20}$, we find, by Lemma 3, that the functions $r_1$ and $r_2$ are infinitely differentiable on the set E, and their differentials at the point $y_0$ with respect to the set E are equal to the corresponding differentials of the functions $\tilde{r}_1$ and $\tilde{r}_2$. Q.E.D.

**Section 5. Proof of the Theorem.** The billiard transformation T constructed for the region G preserves the measure $\nu$ in the phase space M given by the formula $d\nu = \langle n(q), v\rangle d\mu(q, v)$ [4, p. 135]. Since $\langle n(q), v\rangle > 0$ for any element $(q, v) \in M$, the measures $\nu$ and $\mu$ are equivalent, which implies the equivalence of the measures $\mu$ and $\mu T$. Further, the measure $\mu$ is equivalent to the Lebesgue measure in an arbitrary smooth local coordinate system on K, as in a smooth manifold in $\partial G \times S^{n-1}$. It follows, by Lemma 2, that for almost all (with respect to the measure $\mu$) elements x of the set S (see Sec. 4), the elements x and $T^{-1}x$ are simultaneously points of density of the set S in the $\partial G$-section. Let $x \in S$ be an element with the indicated property (it certainly exists if $\mu(S) > 0$). Let $A_0$, $A_1$, and $A_2$ denote the support of x, Tx, and $T^2x = T^{-1}x$, respectively. Through the points $A_1$ and $A_2$ we draw surfaces $L_1$ and $L_2$ as in Sec. 1. We proceed in the analogous way with $A_2$ (instead of $A_0$) taken to be the initial point. The surfaces $\tilde{L}_1$ and $\tilde{L}_2$ will then pass through the points $A_0$ and $A_1$, respectively.

In the neighborhood of the point $A_1$ we choose a $C^\infty$-smooth system of coordinates $(\xi; \xi^n) = (\xi^1, \ldots, \xi^{n-1}; \xi^n)$ so that the $\xi^n$ axis is transverse to the boundary $\partial G$ at the point $A_1$. Then the boundary $\partial G$ and the surfaces $L_1$ and $\tilde{L}_2$ in the neighborhood of the point $A_2$ are given by the equations $\xi^n = f(\xi)$, $\xi^n = f_1(\xi)$, $\xi^n = f_2(\xi)$, respectively, where f is a $C^1$-smooth function and $f_1$ and $f_2$ are $C^\infty$-smooth functions in the neighborhood of $\xi_0 \in \mathbb{R}^{n-1}$, and $f(\xi_0) = f_1(\xi_0) = f_2(\xi_0) = \xi_0{}^n$ (here $(\xi;\xi_0{}^n)$ are the coordinates of the point $A_1$).

It follow from Lemma 4, by the choice of the element x, that there exist measurable sets $E_1$ and $E_2$ in $\mathbb{R}^{n-1}$ that have a point of density at $\xi_0$ and the property that the function f is infinitely differentiable on $E_1$ and $E_2$, where is differentials at $\xi_0$ with respect to $E_1$ and $E_2$ coincide with the analogous differentials of the functions $f_1$ and $f_2$, respectively, at the point $\xi_0$. However, it is obvious that the differentials of f at the point $\xi_0$ with respect to the sets $E_1$ and $E_2$ are the same (since $\xi_0$ is also a point of density for the set $E_1 \cap E_2$), so all differentials of the functions $f_1$ and $f_2$ at $\xi_0$ coincide, i.e., the surfaces $L_1$ and $\tilde{L}_2$ have points of tangency of infinite order at the point $A_1$. This is, however, impossible, since, in the normal cross-section of these surfaces, a two-dimensional plane passing through $A_0$, $A_1$, $A_2$ yields two different second-order curves (by construction, the foci of these curves do not coincide), which cannot have even a fourth-order point of tangency at $A_1$. This contradiction proves the theorem.

**Remark.** It can be shown that the above-noted surfaces $L_1$ and $\tilde{L}_2$ have second-order tangency at the point $A_1$, but not third-order.

## REFERENCES


1. M. R. Rychlik, "Periodic points of the billiard ball map in a convex domain," J. Diff. Geom., **30**, No. 1, 191-205 (1989).
2. L. Stojanov, "Note on the periodic points of the billiard," J. Diff. Geom., **34**, No. 3, 835-837 (1991).
3. G. E. Shilov and B. L. Gurevich, Integrals, Differentials, and Measure [in Russian], Nauka, Moscow (1967).
4. I. P. Kornfel'd, Ya. G. Sinai, and S. V. Fomin, Ergodic Theory [in Russian], Nauka, Moscow (1980).