Proof of Theorem 3. Choose a Riemannian metric on M. Since the critical points of f are nondegenerate, the singularities of grad f are isolated and simple. Thus the index of grad f is 1, at a point where f is either maximum or minimum, or the index of grad f is -1, at a saddle point of f. It follows that $M - s + m$ is equal to the sum of the indices of the singularities of grad f. By the Gauss Bonnet theorem, such a sum does not depend on the chosen metric or on the field grad f, and it is equal to $\chi(M^2)$. □

Remark. Theorem 2 is only a sample of the deep relations established by M. Morse between the topology of differentiable manifolds and the critical points of certain classes of differentiable functions. A beautiful introduction to the subject is J. Milnor [MILN].

EXERCISES

1) Compute the Euler-Poincaré characteristic of
   a) an ellipsoid,
   b) $M = \{(x, y, z) \in \mathbf{R}^3;\ x^2 + y^4 + z^6 = 1\}$.
2) Prove that there exists no Riemannian metric on a torus T such that K is nonzero and does not change sign on T.
3) Let M^2 be a connected compact orientable manifold of dimension two. Prove that the following statements are equivalent (Assume that if $\chi(M^2) = \chi(\overline{M}^2)$ then M^2 is homeomorphic to $\overline{M}^2$):
   a) These exists a nowhere zero differentiable vector field on M^2.
   b) $\chi(M^2) = 0$.
   c) M^2 is homeomorphic to a torus.
4) Let $M^2 \subset \mathbf{R}^3$ be a regular surface in $\mathbf{R}^3$. Assume that M^2 is compact, oriented and not homeomorphic to a sphere. Show that there exist points in M^2 for which the Gaussian curvature is positive, negative and zero.
5) Let M^2 be a connected, compact, oriented Riemannian manifold of dimension two such that the Gaussian curvature K is always positive. Prove that two simple closed geodesics in M^2 have a common point.
6) Let $f: \mathbf{R}^2 \to \mathbf{R}$ be given by (the monkey saddle) $f(x, y) = x^3 - 3xy^2$. Let $p = (0, 0) \in \mathbf{R}^2$. Show that:
   a) p is an isolated critical point of f.
   b) p is a degenerate critical point.
   c) The index of grad f at p is equal to -2.
7) Let $x: M^2 \to \mathbf{R}^3$ be an immersion of a two-dimensional differentiable manifold M^2 into $\mathbf{R}^3$(i.e., a surface in $\mathbf{R}^3$), and let $h_\nu: M \to \mathbf{R}$ be the *height function*, $h_\nu(p) = \langle x(p), \nu\rangle, p \in M$, of x relative to a fixed unit vector $\nu \in \mathbf{R}^3$($h_\nu(p)$ measures the "height" of $x(p)$ relative to a plane through the origin and perpendicular to ν.).

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