$$K(p) = \lim_{D \to p} \frac{\varphi}{\text{area } D},$$

that is, the Gaussian curvature at p measures how different from the identity is parallel transport along small circles about p.

EXERCISES

1) *(The flat torus).* Let $f: \mathbf{R}^2 \to \mathbf{R}^4$ be given by

$$f(x,y) = (\cos x, \sin x, \cos y, \sin y), \quad (x,y) \in \mathbf{R}^2.$$

Prove that:

a) f is an immersion and $f(\mathbf{R}^2)$ is homeomorphic to a torus,

b) The frame $e_1 = \frac{\partial f}{\partial x}$, $e_2 = \frac{\partial f}{\partial y}$ in $f(\mathbf{R}^2) \subset \mathbf{R}^4$ is orthonormal in the metric of $f(\mathbf{R}^2)$ induced by $\mathbf{R}^4$. Compute η^1, η^2, and the connection form.

c) The Gaussian curvature of the induced metric is identically zero.

2) *(The hyperbolic plane).* Let H^2 be the upper half-plane, that is,

$$H^2 = \{(x,y) \in \mathbf{R}^2; y > 0\}$$

Consider in H^2 the following inner product: If $(x,y) \in H^2$ and $u, v \in T_pH^2$, then

$$\langle u, v \rangle_p = \frac{u \cdot v}{y^2},$$

where $u \cdot v$ is the canonical inner product of $\mathbf{R}^2$. Prove that this is a Riemannian metric in H^2 whose Gaussian curvature is $K \equiv -1$; with this Riemannian metric H^2 is called the hyperbolic plane.

Hint: Choose the orthonormal frame $e_1 = y a_1$, $e_2 = y a_2$, where $\{a_1, a_2\}$ is the canonical frame of $\mathbf{R}^2$.

3) Let M^2 be a Riemannian manifold of dimension two. Let $f: U \subset \mathbf{R}^2 \to M$ be a parametrization of M^2 such that $f_u = df(\frac{\partial}{\partial u})$ and $f_v = df(\frac{\partial}{\partial v})$, $(u,v) \in U$, are orthogonal. Set $E = \langle f_u, f_u \rangle$ and $G = \langle f_v, f_v \rangle$. Choose an orthonormal frame $e_1 = f_u/\sqrt{E}$, $e_2 = f_v/\sqrt{G}$ in U. Show that:

a) The associated coframe is given by

$$\eta^1 = \sqrt{E}du, \quad \eta^2 = \sqrt{G}dv.$$

b) The connection form is given by

$$\eta_1^2 = -\frac{(\sqrt{E})_v}{\sqrt{G}}du + \frac{(\sqrt{G})_u}{\sqrt{E}}dv.$$

Hint: Use the fact that $\eta_1^2(e_i) = d\eta^i(e_1, e_2)$, $i = 1, 2$.

c) The Gaussian curvature of M^2 is

$$K = -\frac{1}{\sqrt{EG}}\left\{\left(\frac{(\sqrt{E})_v}{\sqrt{G}}\right)_v + \left(\frac{(\sqrt{G})_u}{\sqrt{E}}\right)_u\right\}.$$

4) Let $S^2 = \{(x,y,z) \in \mathbf{R}^3; x^2+y^2+z^2 = 1\}$. Prove that there exists no differentiable nonzero vector field X on S^2.
Hint: Assume the existence of such a field X. Let $e_1 = X/|X|$ e consider the orthonormal oriented frame $\{e_1.e_2\}$. Then $d\omega_{12} = -K\omega_1 \wedge \omega_2 = -\sigma$, hence

$$\text{area } S^2 = \int_{S^2} \sigma = -\int_{S^2} d\omega_{12} = -\int_{\partial S^2} \omega_{12} = 0,$$

which is a contradiction.

5) Consider $\mathbf{R}^2$ with the following inner product: If $p = (x,y) \in \mathbf{R}^2$ and $u, v \in T_p\mathbf{R}^2$, then

$$\langle u, v\rangle_p = \frac{u \cdot v}{(g(p))^2},$$

where $u \cdot v$ is the canonical inner product of $\mathbf{R}^2$ and $g: \mathbf{R}^2 \to \mathbf{R}$ is a differentiable positive function. Prove that the Gaussian curvature of this metric is

$$K = g(g_{xx} + g_{yy}) - (g_x^2 + g_y^2).$$

6) Let $M^2 \subset \mathbf{R}^3$ be a surface with the induced metric. Let $p \in M^2$, $x \in T_pM^2$ and Y be a vector field tangent to M^2 . Show that

$$(\nabla_x Y)(p) = \text{projection onto } T_pM \text{ of } \left(\frac{dY(\alpha(s))}{ds}\right)(0),$$

where $\alpha: I \to M$ is a differentiable curve, $s \in I$, and $\frac{dY}{ds}$ is the usual derivative of vectors in $\mathbf{R}^3$. Conclude that a curve $\gamma(s)$ in M, parametrized by the arc length s, is a geodesic in M if and only if the "acceleration" vector $\frac{d^2\gamma}{ds^2}$ in $\mathbf{R}^3$ is everywhere perpendicular to M.

7) Let $S^2 = \{(x,y,z) \in \mathbf{R}^3; x^2+y^2+z^2 = 1\}$ be the unit sphere with the metric induced from $\mathbf{R}^3$. Show that:
   a) The geodesics of S^2 are its great circles,
   b) The antipodal map $A: S^2 \to S^2$ given by $A(x,y,z) = (-x,-y,-z)$ is an isometry,
   c) The projective plane $P^2(\mathbf{R})$ (cf. Example 7 of Chapter 2) can be given a Riemannian metric such that the canonical projection $\pi: S^2 \to P^2(\mathbf{R})$ is a local isometry (that is, each $p \in S^2$ has a neighborhood V such that the restriction π/V is an isometry).

8) Let M^2 be a Riemannian manifold (of dimension two). The goal of the exercise is to show that the Gaussian curvature K of M is identically zero if and only if M is locally euclidean, that is, there exist local coordinates