Solution of Homework assignment 2

Problem 1

a) Let $Q_t = P\circ \overrightarrow{\exp}\int_0^t V_\tau d\tau \circ P^{-1}$

Then $\frac{d}{dt}Q_t = P\circ \frac{d}{dt}\overrightarrow{\exp}\int_0^t V_\tau d\tau\circ P^{-1} = P\circ\overrightarrow{\exp}\int_0^t V_\tau d\tau \circ V_t\circ P^{-1} =$

$= P\circ\overrightarrow{\exp}\int_0^t V_\tau d\tau\circ P^{-1})\circ (P\circ V_t\circ P^{-1}) = Q_t\circ \mathrm{Ad}P\, V_t$, i.e

$$\left.\begin{array}{l}\frac{d}{dt}Q_t = Q_t\circ \mathrm{Ad}P\,V_t\\ Q_0=\mathrm{Id}\end{array}\right\} \underset{\text{uniqueness of a Cauchy problem}}{\Longrightarrow} Q_t = \overrightarrow{\exp}\int_0^t \mathrm{Ad}P\,V_\tau \, d\tau \quad \blacksquare$$

b) By variational formula

$$e^{t(V+\varepsilon W)} = \overrightarrow{\exp}\int_0^t e^{\tau\,\mathrm{ad}V}\varepsilon W\, d\tau\circ e^{tV}$$

From Volterra asymptotic expansion

$$\overrightarrow{\exp}\int_0^t e^{\tau\,\mathrm{ad}V}\varepsilon W\, d\tau = \mathrm{Id} + \varepsilon\int_0^t e^{\tau\,\mathrm{ad}W} W\, d\tau +$$

$$+ \varepsilon^2\int_0^t\int_0^{\tau_1} e^{\tau_2\,\mathrm{ad}V}W\circ e^{\tau_1\,\mathrm{ad}V}W\, d\tau_2\, d\tau_1 + O(\varepsilon^3),$$ which implies the required asymptotics.

c) Since $\int_0^T e^{\tau_1\,\mathrm{ad}V}W\, d\tau_1 = 0$ then

$$\int_0^T\int_0^T e^{\tau_2\,\mathrm{ad}V}W\circ e^{\tau_1\,\mathrm{ad}V}W\, d\tau_2\, d\tau_1 = \int_0^T e^{\tau_2\,\mathrm{ad}V}W\, d\tau_2\circ \int_0^T e^{\tau_1\,\mathrm{ad}V}W\, d\tau_1 = 0$$

$$\Downarrow$$

$$\int_0^T\int_0^{\tau_1} e^{\tau_2\,\mathrm{ad}V}W\circ e^{\tau_1\,\mathrm{ad}V}W\, d\tau_2\, d\tau_1 = \int_0^T\int_0^{\tau_1} e^{\tau_2\,\mathrm{ad}V}W\circ e^{\tau_1\,\mathrm{ad}V}W\, d\tau_2\, d\tau_1 -$$

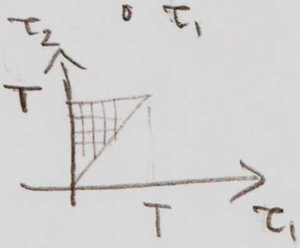
$$-\frac{1}{2}\int_0^T\int_0^T e^{\tau_2\,\mathrm{ad}V}W\circ e^{\tau_1\,\mathrm{ad}V}W\, d\tau_2\, d\tau_1 = \frac{1}{2}\Bigg(\int_0^T\int_0^{\tau_1} e^{\tau_2\,\mathrm{ad}V}W\circ e^{\tau_1\,\mathrm{ad}V}W\, d\tau_2\, d\tau_1 -$$

$$-\int_0^T\int_{\tau_1}^T e^{\tau_2\,\mathrm{ad}V}W\circ e^{\tau_1\,\mathrm{ad}V}W\, d\tau_2\, d\tau_1\Bigg) = (*)$$

(in the last formula we used that $\int_0^T\int_0^T e^{\tau_2\,\mathrm{ad}V}W\circ e^{\tau_1\,\mathrm{ad}V}W\, d\tau_1 d\tau_2 = \int_0^T\int_0^{\tau_1} e^{\tau_2\,\mathrm{ad}V}W\circ e^{\tau_1\,\mathrm{ad}V}W\, d\tau_2 d\tau_1 + \int_0^T\int_{\tau_1}^T e^{\tau_2\,\mathrm{ad}V}W\circ e^{\tau_1\,\mathrm{ad}V}W\, d\tau_2 d\tau_1$

Finally, change the order of integration in the last integral of (*):

$$\int_0^T\int_{\tau_1}^T e^{\tau_2 \,\mathrm{ad} V} W \circ e^{\tau_1 \,\mathrm{ad} V} W \, d\tau_2 \, d\tau_1 = \int_0^T\int_0^{\tau_2} e^{\tau_2 \,\mathrm{ad} V} W \circ e^{\tau_1 \,\mathrm{ad} V} W \, d\tau_1 \, d\tau_2 =$$



$$\underset{\text{(exchange the role of } \tau_1 \text{ and } \tau_2)}{=} \int_0^T\int_0^{\tau_1} e^{\tau_1 \,\mathrm{ad} V} W \circ e^{\tau_2 \,\mathrm{ad} V} W \, d\tau_2 \, d\tau_1$$

Substituting this to (*) we get

$$\int_0^T\int_0^{\tau_1} e^{\tau_2 \,\mathrm{ad} V} W \circ e^{\tau_1 \,\mathrm{ad} V} W \, d\tau_2 \, d\tau_1 = \frac{1}{2}\int_0^T\int_0^{\tau_1} \left(e^{\tau_2 \,\mathrm{ad} V} W \circ e^{\tau_1 \,\mathrm{ad} V} W - \right.$$

$$\left. - e^{\tau_1 \,\mathrm{ad} V} W \circ e^{\tau_2 \,\mathrm{ad} V} W \right) d\tau_2 \, d\tau_1 = \frac{1}{2}\int_0^T\int_0^{\tau_1} \left[e^{\tau_2 \,\mathrm{ad} V} W, e^{\tau_1 \,\mathrm{ad} V} W \right] d\tau_2 \, d\tau_1$$

$\blacksquare$

Problem 2

a) $[V_A, V_B] = dV_B V_A - dV_A V_B$

$dV_B = B$, $dV_A = A$ $\Rightarrow$

$[V_A, V_B](x) = BAx - ABx = (BA - AB)x = V_{BA-BA}(x)$

b) First find $Lie_x(F)$ for any x

$$[V_A, V_B](x) = V_{BA-AB}(x) =$$

$$= (BA - AB)x = \left(\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \right) x =$$

$$= \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -x_3 \\ 0 \\ x_1 \end{pmatrix}.$$ Let $C = BA - AB$

If $x \neq 0$, $\dim \operatorname{span}(Ax, Bx, Cx) = 2$ because the rank of the corresponding matrix of columns $\begin{pmatrix} x_2 & 0 & x_3 \\ -x_1 & x_3 & 0 \\ 0 & -x_2 & -x_1 \end{pmatrix}$ is equal to 2

$\left(\begin{vmatrix} x_2 & 0 & x_3 \\ -x_1 & x_3 & 0 \\ 0 & -x_2 & -x_1 \end{vmatrix} = -x_1x_2x_3 + x_1x_2x_3 = 0 \right.$ but if one of $x_i \neq 0$ there exists non-zero 2x2 minor $\left. \right)$

Besides, all other Lie brackets are in span (Ax, Bx, Cx),

because $CB - BC = A$ and $AC - CA = B$.

or equivalently $[V_B, V_C] = V_A$ and $[V_C, V_A] = V_B$

(actually the field V_A, V_B, V_C are the basis of 3-dimensional Lie algebra $\sim so(3)$)

Therefore for $x \neq 0$ $\quad \mathcal{L}ie_x(\mathcal{F}) = \text{span}\langle Ax, Bx, Cx\rangle$

$$\dim \mathcal{L}ie_x(\mathcal{F}) = 2$$

The vector fields V_A and V_B are linear $\Rightarrow$ real analytic $\Rightarrow$

$\mathcal{L}ie_x(\mathcal{F}) = T_x O_{\bar{x}}$ for any $x \in O_{\bar{x}}$, i.e. the orbits O_x for $x \neq 0$ are 2-dimensional immersed submanifold of $\mathbb{R}^3$

Since A and B are antisymmetric e^{tA} and e^{tB} are orthogonal (as a matter of fact e^{tA} is the rotation around x_3-axis (by the angle t) and e^{tB} is the rotation around x_1-axis (by the angle t)) $\Rightarrow$ the orbit O_x belongs to the sphere of radius $\|x\|$. Since O_x is 2-dimensional, O_x is an open connected subset of this sphere, therefore it coincides with the sphere.

If $x = 0$ then $O_x = \{0\}$, because $x = 0$ is the stationary point of both V_A and V_B.

<u>Rem</u> Actually, to solve this problem you do not need any theory: Since the flows of V_A and V_B are orthogonal transformations, then O_{x_0} lies in the sphere of radius $\|x_0\|$ and it is easy to understand how to reach any point of this sphere from x_0 by a concatenation of rotations around x_3-axis and x_1-axis.

c) Let us calculate $Lie_x(F)$ for any x

$$[V_A, V_B](x) = V_{BA-AB}(x) = (BA-AB)(x) = \left(\begin{pmatrix}1&0&0\\0&-1&0\\0&0&1\end{pmatrix}\begin{pmatrix}0&1&0\\-1&0&0\\0&0&0\end{pmatrix} - \right.$$

$$\left. - \begin{pmatrix}0&1&0\\-1&0&0\\0&0&0\end{pmatrix}\begin{pmatrix}1&0&0\\0&-1&0\\0&0&1\end{pmatrix}\right)x = \left(\begin{pmatrix}0&1&0\\1&0&0\\0&0&0\end{pmatrix} - \begin{pmatrix}0&-1&0\\-1&0&0\\0&0&0\end{pmatrix}\right)x = \begin{pmatrix}0&2&0\\2&0&0\\0&0&0\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} =$$

$$= \begin{pmatrix}2x_2\\2x_1\\0\end{pmatrix} \qquad \text{Let } C = BA-AB$$

· Consider the matrix of columns of $V_A(x), V_B(x), V_C(x)$, i.e $\begin{pmatrix}x_2 & x_1 & x_2\\ -x_1 & -x_2 & x_1\\ 0 & x_3 & 0\end{pmatrix}$

Its determinant is equal to $-2x_1x_2x_3$

So if $x_1x_2x_3 \neq 0$, then $\dim Lie_x(F) = 3$ (1)

Now continue to take brackets at points with $x_1x_2x_3 = 0$

$$[V_C, V_B]x = \left(\begin{pmatrix}1&0&0\\0&-1&0\\0&0&1\end{pmatrix}\begin{pmatrix}0&1&0\\1&0&0\\0&0&0\end{pmatrix} - \begin{pmatrix}0&1&0\\1&0&0\\0&0&0\end{pmatrix}\begin{pmatrix}1&0&0\\0&-1&0\\0&0&1\end{pmatrix}\right)x$$

$$= \left(\begin{pmatrix}0&1&0\\-1&0&0\\0&0&0\end{pmatrix} - \begin{pmatrix}0&-1&0\\1&0&0\\0&0&0\end{pmatrix}\right)x = 2V_A(x) \text{ - nothing new}$$

$$[V_C, V_A]x = \left(\begin{pmatrix}0&1&0\\-1&0&0\\0&0&0\end{pmatrix}\begin{pmatrix}0&1&0\\1&0&0\\0&0&0\end{pmatrix} - \begin{pmatrix}0&1&0\\1&0&0\\0&0&0\end{pmatrix}\begin{pmatrix}0&1&0\\-1&0&0\\0&0&0\end{pmatrix}\right)x$$

$$= \left(\begin{pmatrix}1&0&0\\0&-1&0\\0&0&0\end{pmatrix} - \begin{pmatrix}-1&0&0\\0&1&0\\0&0&0\end{pmatrix}\right)x = 2\begin{pmatrix}x_1\\-x_2\\0\end{pmatrix}$$

The matrix of columns of $V_A(x), V_B(x), [V_C, V_A](x)$ is

$\begin{pmatrix}x_2 & x_1 & x_1\\ -x_1 & -x_2 & -x_2\\ 0 & x_3 & 0\end{pmatrix}$ and its determinant $= x_3(x_2^2 - x_1^2) \Rightarrow$

If $x_3 \neq 0$ and $x_1^2 - x_2^2 \neq 0$ $\quad \dim Lie_x(F) = 3$ (2)

Combining (1) and (2) we get that if $x_3 \neq 0$ and $x_1^2 + x_2^2 \neq 0$ then $\dim Lie_x(F) = 3$

Now consider the case $x_3 = 0$. In this case $V_A(x)$ and $V_B(x)$ are tangent to the plane $x_3 = 0 \Rightarrow$ if x belongs to this plane the orbit O_x is contained in this plane. On the other hand, if x is not the origin, then $V_A(x)$ and $V_B(x)$ are linearly independent $\Rightarrow \dim Lie_x(F) = 2$ if $x_3 = 0$ and $x_1^2 + x_2^2 \neq 0$.

The orbit of the origin is the origin (since $V_A(0) = V_B(0) = 0$)

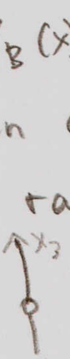
So, if $x_3 = 0$ and $x_1^2 + x_2^2 \neq 0$ the orbit O_x is an open connected subset of $\{x_3 = 0\} \setminus$ the origin. Since $\{x_3 = 0\} \setminus$ the origin is connected then $O_x = \{x_3 = 0\} \setminus \{\text{the origin}\}$

x_3

$\Rightarrow x_1 x_2$-plane

If $x_1^2 + x_2^2 = 0$ then $V_A(x)$ and $V_B(x)$ are parallel to the x_3-axis $\Rightarrow$ If $x_1^2 + x_2^2 = 0$, $x_3 \neq 0$, then O_x is a subset of x_3-axis $\setminus \{\text{the origin}\}$

Besides in this case $V_B(x) \neq 0 \Rightarrow \dim L_A(x) = 1 \Rightarrow$ the orbit O_x is an open connected subset of x_3-axis $\setminus \{\text{the origin}\}$, i.e. O_x is the positive ray if $x_3 > 0$ and the negative ray if $x_3 < 0$.



x_3

Finally for all points with $x_1^2 + x_2^2 \neq 0$ and $x_3 \neq 0$ the orbit is 3-dimensional and it is an open connected subset of $R^3 \setminus \{x_3 = 0\} \cup \{x_3\text{-axis}\}$. Therefore

If $x_3 > 0$, $x_1^2 + x_2^2 \neq 0$, then O_x is $\{\text{the upper half-space}\} \setminus \{\text{positive ray of } x_3\text{-axis}\}$

If $x_3 < 0$, $x_1^2 + x_2^2 \neq 0$ then O_x is $\{\text{the lower half space}\} \setminus \{\text{negative ray of } x_3\text{-axis}\}$

<u>Conclusion:</u>

$x = (x_1, x_2, x_3)$	The orbit
$x_3 \neq 0$, $x_1^2 + x_2^2 \neq 0$	a half-space $\setminus \{x_3\text{-axis}\}$ - 3 dim
$x_3 = 0$, $x_1^2 + x_2^2 \neq 0$	$x_1 x_2$-plane $\setminus$ the origin - 2-dim.
$x_3 \neq 0$, $x_1^2 + x_2^2 = 0$	positive or negative ray of x_3-axis - 1-dim
the origin	the origin - 0-dim