Homework assignment 3 Solutions, MATH 666, Fall 11

First both problems in this set are on the application of the following proposition proved in class:

Proposition 1 Consider an affine control system

$$\dot{q} = f_0(q) + \sum_{i=1}^{m} u_i f_i(q) \quad (u_1, \dots, u_m) \subset U \subset \mathbb{R}^n$$
$$q \in M$$

Assume that:

1) $Lie_q(f_0, f_1, \dots, f_m) = T_q M \quad \forall_q$;
2) $0 \in \operatorname{Int}(\operatorname{conv} U)$;
3) The drift f_0 is weakly Poisson stable.

Then the system is controllable.

Problem 1 An inertia axis is by definition a line of eigenvectors of the inertia operator A. Assume that I_1 and I_2 are the eigenvalues of A corresponding to l_1 and l_2 respectively.

i) The case $I_1 \neq I_2$. In this case $l_1 \perp l_2$, i.e. $e_1 \perp e_2$. So we can take e_1, e_2 to (in the directions of e_1 and e_2, respectively) be unit vectors and we can complete the pair (e_1, e_2) to the orthonormal basis, in which the inertia operator A is diagonal.

If (μ_1, μ_2, μ_3) are the coordinates (in $\mathbb{R}^3$) w.r.t. this basis, then in this coordinates the control system (1) has the form

$$\begin{cases} \dot{\mu}_1 = a_1 \mu_2 \mu_3 + u_1 \alpha_1 \\ \dot{\mu}_2 = a_2 \mu_1 \mu_3 + u_2 \alpha_2 \\ \dot{\mu}_3 = a_3 \mu_1 \mu_2 \end{cases}, \quad u_1, u_2 \in \{-1, 1\}$$

where u_i are constants, $a_3 = \frac{1}{I_2} - \frac{1}{I_1} \neq 0$, $\alpha_i = \|e_i\| \neq 0$

Let us verify the conditions of Proposition 1:

Condition 1) In the chosen coordinates

$$f_0 = a_1 \mu_2 \mu_3 \frac{\partial}{\partial \mu_1} + a_2 \mu_1 \mu_3 \frac{\partial}{\partial \mu_2} + a_3 \mu_1 \mu_2 \frac{\partial}{\partial \mu_3}$$

$$f_1 = \alpha_1 \frac{\partial}{\partial \mu_1}, \quad f_2 = \alpha_2 \frac{\partial}{\partial \mu_2}$$

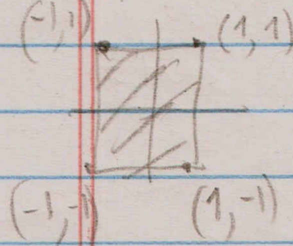
Then $$\left[\frac{1}{\alpha_1} f_1, f_0\right] = a_2 \mu_3 \frac{\partial}{\partial \mu_1} + a_3 \mu_2 \frac{\partial}{\partial \mu_3}$$

$$\left[\frac{1}{\alpha_2} f_2, \left[\frac{1}{\alpha_1} f_1, f_0\right]\right] = a_3 \frac{\partial}{\partial \mu_3}$$

Since in the considered case $a_3 \neq 0$,

$$\operatorname{span}\left\{f_1(q), f_2(q), \left[\frac{1}{\alpha} f_2, \left[\frac{1}{\alpha_1} f_1, f_0\right]\right](q)\right\} = T_q \mathbb{R}^3 \text{ for any } q.$$

So, $\mathrm{Lie}_q(f_0, f_1, f_2) = T_q \mathbb{R}^3$ for any q



Condition 2) of Proposition 1 holds:

conv U = the square with vertices $(1,1), (-1,1), (-1,-1), (1,1)$ so $(0,0) \in \mathrm{int}(\mathrm{conv}\, U)$.

Condition 3) holds because $\mathrm{div} f_0 = 0$ and the balls around the origin are invariant sets of f_0 (as discussed in class after proving Theorem 8.3 of the textbook)

So, if $I_1 \neq I_2$, then control system (1) is controllable.

ii) Now assume that $I_1 = I_2$. Then l_1 is not necessary orthogonal to l_2 but you can choose an orthogonal basis $(\tilde{e}_1, \tilde{e}_2, \tilde{e}_3)$ of $\mathbb{R}^3$ diagonalizing the inertia operator A s.t $l_1 = \mathbb{R}\tilde{e}_1$ and $\mathrm{span}(l_1, l_2) = \mathrm{span}(\tilde{e}_1, \tilde{e}_2)$. If (μ_1, μ_2, μ_3) is the coordinates in $\mathbb{R}^3$ w.r.t. the basis $(\tilde{e}_1, \tilde{e}_2, \tilde{e}_3)$ then the system (1) has the following form in this basis:

$$\begin{cases} \dot{\mu}_1 = a_1 \mu_2 \mu_3 + \alpha_1 u_1 + \alpha_{21} u_2 \\ \dot{\mu}_2 = a_2 \mu_2 \mu_3 + \alpha_{22} u_2 \\ \dot{\mu}_3 = 0 \quad \left(\text{Because } a_3 = \frac{1}{I_2} - \frac{1}{I_1} = 0\right) \end{cases}$$

Since $\dot{\mu}_3 \equiv 0$ then μ_3 is the integral of the control system (1). So the attainable set of the point $(\bar{\mu}_1, \bar{\mu}_2, \bar{\mu}_3)$ belong to the plane $\mu_3 = \bar{\mu}_3$ and the system is not controllable

Conclusion System (1) is controllable $\Leftarrow$ $\boxed{I_1 \neq I_2}$

Problem 2 a) Let us verify condition 1 of Proposition 1

Here $f_0(E) = EA$, $f_1(E) = EB$, where

$$A = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ and } B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$$

Remark 1 (clarification). Let $M_{3\times3}$ be the set of all 3×3 matrices. Then $M_{3\times3}$ can be identified with $\mathbb{R}^9$ (the tuple of entries define coordinates in $M_{3\times3}$). Then $v_A(E) := EA$ can be considered as a vector field (components of this vector field at E are equal to the corresponding entries of the matrix EA)

$SO(3)$ is a submanifold of $M_{3\times3}$
Besides, the tangent space $T_E SO(3)$ to $SO(3)$ at E is $T_E SO(3) = \{E\Omega : \Omega \text{ is antisymmetric}\}$
(It is the standard fact $\underbrace{\text{and it is actually}}_{\text{(also discussed in class)}}$ used in the formulation of the problem: the right hand side of (2) is tangent to $SO(3)$, therefore (2) can be indeed considered as a control system on $SO(3)$)
So, if A is antisymmetric then the restriction of $V_A(E)(=EA)$ to $SO(3)$ defines the vector field on $SO(3)$ $\boxtimes$

Let us calculate $Lie_E(f_0, f_1)$

$$[f_0, f_1](E) = [EA, EB] = d(EB)EA - d(EA)EB$$

How to calculate $d(EB)EA$?

Recall that if $x \mapsto Lx$ is a linear map then its differential is equal to the map itself, i.e

$$\underbrace{dL(x)\,y}_{\substack{\text{the differential}\\ \text{of } L \text{ at } x \text{ acting}\\ \text{on the vector } y}} = Ly$$

The map $E \to EB$ on $M_{3\times3}$ ($\cong \mathbb{R}^9$) is linear (each entry of EB is linear w.r.t. the entries of E) $\Rightarrow$

$$d(EB)Y = YB \quad \text{for any } Y \in T_E M_{3\times3} \cong M_{3\times3}$$

$$\Downarrow$$

$$d(EB)EA = (EA)B = EAB$$

(Here we also use that the differential of a restriction of a map to a submanifold is equal to the restriction of the differential of this map to the tangent space of this submanifold: we restrict the map $E \to EB$ to $SO(3)$)

In the same way $d(EA)EB = EBA$

So $[EA, EB] = E(AB - BA)$

$$AB - BA = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$\dim \operatorname{span}(A, B, AB-BA) = 3 \Rightarrow$

$\dim \operatorname{span}(EA, EB, E(AB-BA)) = 3$

$\dim SO(3) = 3 \Rightarrow Lie_E(f_0, f_1) = T_E SO(3)$

Condition 2 holds because $conv\{-1,1\} = [-1,1]$ and $Int\,[-1,1] = (-1,1)$ and $0 \in (-1,1)$.

<u>Condition 3</u> $\underbrace{e^{tf_0}}_{\text{the flow of the vector field } f_0} = Ee^{tA}$, where e^{tA} is the usual matrix exponent

Let us prove that all trajectories of vector field f_0 are periodic. This will imply that f_0 is Poisson stable (and therefore weakly Poisson stable)

Note that e^{tA} is the rotation by angle t around z-axis (counterclockwise as seen from the top):

$$e^{tA} = \begin{pmatrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 0 \\ 0 & 0 & 1 \end{pmatrix}$$ So $t \to e^{tA}$ is periodic with the period 2π. The same holds for $\forall E e^{tA}$ q.e.d.

So all 3 conditions of Proposition 1 holds $\Rightarrow$ the system is controllable.

<u>Remark 2</u> The same conclusion about periodicity of each trajectory is true for any vector field $V_\Omega(E) = E\Omega$ for Ω being antisymmetric, because as discussed in class $e^{t\Omega}$ generates the

rotation with constant angular velocity around certain axis; if $\Omega = \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix}$ then it is around the axis $(\omega_1, \omega_2, \omega_3)$ with the angular velocity $\sqrt{\omega_1^2+\omega_2^2+\omega_3^2}$.

b) To handle this problem one can modify slightly Proposition 1':

Instead of cond 2) and 3) we can assume that $f_0 + \bar{u}_1 f_1 + \ldots + \bar{u}_m f_m$ is weakly Poisson stable for some control

$$(\bar{u}_1, \ldots, \bar{u}_m) \in \mathrm{Int}(\mathrm{conv}\, U)$$

Indeed $\dot{q} = f_0 + u_1 f_1 + \ldots + u_m f_m =$

$$= \underbrace{f_0 + \bar{u}_1 f_1 + \ldots + \bar{u}_m f_m}_{\bar{f}_0} + \underbrace{(u_1 - \bar{u}_1)}_{v_1} f_1 + \ldots + \underbrace{(u_m - \bar{u}_m)}_{v_m} f_m$$

So by shift of the control space,

$v_i = u_i - \bar{u}_i \quad 1 \le i \le m$, we get that the origin is in the interior of the convexification of the new (shifted) control space. So we are reduced to the conditions of Proposition 1.

So in b) take $\bar{u} = \frac{3}{2}$ for example

$\bar{f}_0 = f_0 + \frac{3}{2} f_1$.

Then condition 1 holds from a)

$\frac{3}{2} \in \text{int conv}\{1,2\}$ and $\bar{f}_0$ is Poisson stable

by Remark 2. So, we have controllability in b) as well.