Homework Assignment 2 in Geometric Control Theory, MATH666

due to Oct 7, 2011

Problem 1 In this problem we assume that all vector fields satisfy conditions 1)-3) of subsection 2.4.1 of the textbook so that the existence and uniqueness theorem for the corresponding initial value problems (the Cauchy problem) holds. You can use freely the formulas from Chronological Calculus that were proved in class.

a) Show that if $P \in \mathrm{Diff}(\mathrm{M})$ and V_t is a nonautonomous vector field on M, then

$$P \circ \overrightarrow{\exp} \int_0^t V_\tau \, d\tau \circ P^{-1} = \overrightarrow{\exp} \int_0^t (\mathrm{Ad}P \, V_\tau) \, d\tau.$$

b) Assume that V and W are two autonomous vector fields in M and the flows e^{tV} and e^{tW} are defined for $0 \leq t \leq t_1$. Prove that for any such t

$$e^{t(V+\varepsilon W)} = e^{tV} + \varepsilon \int_0^t e^{\tau \mathrm{ad} V} W \; d\tau \circ e^{tV} + \varepsilon^2 \int_0^t \int_0^{\tau_1} e^{\tau_2 \mathrm{ad} V} W \circ e^{\tau_1 \mathrm{ad} V} W \; d\tau_2 d\tau_1 \circ e^{tV} + O(\varepsilon^3)$$

c) Let V and W be as in the previous item and assume in addition that for some time $T \in (0, t_1]$ we have

$$\int_0^T e^{\tau_1 \mathrm{ad} V} W d\tau_1 = 0.$$

Prove that

$$e^{T(V+\epsilon W)} = e^{TV} + \frac{\varepsilon^2}{2} \int_0^T \int_0^{\tau_1} \left[e^{\tau_2 \mathrm{ad} V} W, e^{\tau_1 \mathrm{ad} V} W \right] d\tau_2 d\tau_1 \circ e^{TV} + O(\varepsilon^3).$$

Problem 2 Assume that $M = \mathbb{R}^n$ and for any $n \times n$ matrix A denote by V_A the linear vector field defined by this matrix, i.e. $V_A(x) = Ax$, for any $x \in \mathbb{R}^n$.

a) Prove that $[V_A, V_B] = V_{BA-AB}$ (here AB is just the matrix multiplication).

b) Assume that $n = 3$ and

$$A = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}.$$

Describe all orbits of $\mathcal{F} = \{V_A, V_B\}$.

c) Change the matrix B in the previous problem by

$$B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

and describe all orbits of $\mathcal{F} = \{V_A, V_B\}$.