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NUROWSKI'S CONFORMAL STRUCTURES FOR (2,5)-DISTRIBUTIONS VIA DYNAMICS OF ABNORMAL EXTREMALS

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Abstract. As was shown recently by P. Nurowski, to any rank 2 maximally nonholonomic vector distribution on a 5-dimensional manifold M one can assign the canonical conformal structure of signature $(3,2)$. His construction is based on the properties of the special 12-dimensional coframe bundle over M, which was distinguished by E. Cartan during his famous construction of the canonical coframe for this type of distributions on some 14-dimensional principal bundle over M. The natural question is how "to see" the Nurowski conformal structure of a $(2,5)$-distribution purely geometrically without the preliminary construction of the canonical frame. We give rather simple answer to this question, using the notion of abnormal extremals of $(2,5)$-distributions and the classical notion of the osculating quadric for curves in the projective plane. Our method is a particular case of a general procedure for construction of algebra-geometric structures for a wide class of distributions, which will be described elsewhere. We also relate the fundamental invariant of $(2,5)$-distribution, the Cartan covariant binary biquadratic form, to the classical Wilczynski invariant of curves in the projective plane.

1. Introduction

1.1. **Statement of the problem.** The following rather surprising fact was discovered by P. Nurowski recently in [8]: any rank 2 maximally nonholonomic vector distribution on a 5-dimensional manifold M possesses the canonical conformal structure of signature $(3,2)$. His construction is based on the Cartan famous paper [4], where the canonical coframe for this type of distributions on some 14-dimensional principal bundle P over M was constructed. In the modern terminology Cartan constructed the Cartan normal connection, valued on the split real form $\widetilde{G}_2$ of the exceptional Lie algebra G_2, which is the algebra of infinitesimal symmetries of the most symmetric distribution of the considered type. It is well-known that the algebra $\tilde{G}_2$ can be naturally imbedded into the algebra $so(4,3)$, which in turn is isomorphic to the conformal algebra of signature $(3,2)$. Motivated by this fact, P. Nurowski noticed that a simple quadratic expression, involving only the 1-forms from the mentioned canonical coframe, which annihilate the fibers of P, is transformed conformally along the fibers of P and therefore defines the canonical conformal structure on M. In his construction he uses actually not the bundle P but the special 12-dimensional coframe bundle, to which E. Cartan reduces the bundle of all coframes in the first step of his reduction-prolongation procedure (see also section 4 below). Note that this reduction is highly nontrivial and purely algebraic in nature. We asked ourselves, how "to see" the field of cones, defining the Nurowski conformal structure of a $(2,5)$-distribution, purely geometrically without exploiting the Cartan equivalence method, the bundles of so high (with compare to M) dimensions and the properties of the group of symmetries of the most symmetric model? We give quite elementary answer

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to this question, using the dynamics of so-called abnormal extremals (or singular curves) of $(2,5)$-distributions, living in some 6-dimensional bundle over M (so that the fibers are just one-dimensional), and the classical notion of the osculating quadratic cones for curves in the projective plane.

1.2. **The main constructions.** We start our construction with the description of characteristic curves of $(2,5)$-distribution D. Let D^l be the l-th power of the distribution D, i.e., $D^l = D^{l-1} + [D, D^{l-1}]$, $D^1 = D$. The tuple $\big(\dim D(q), \dim D^2(q), \dim D^3(q), \dots\big)$ is called the *small growth vector of D at q*. Throughout this paper we will consider $(2,5)$-distributions with the small growth vector $(2,3,5)$ at any point. Let $\pi : T^*M \mapsto M$ be the canonical projection. For any $\lambda \in T^*_qM$, $q \in M$, let $\varsigma_\lambda(\cdot) = \lambda(\pi_*\cdot)$ be the canonical Liouville form and $\sigma = d\varsigma$ be the standard symplectic structure on T^*M. Denote by $(D^l)^\perp \subset T^*M$ the annihilator of the lth power D^l, namely

$$(D^l)^\perp = \big\{\lambda \in T^*M : \ \langle \lambda, v\rangle = 0 \ \forall\, v \in D^l\big(\pi(\lambda)\big)\big\}. \tag{1.1}$$

Since the submanifold $(D^2)^\perp$ has odd codimension in T^*M, the kernels of the restriction $\sigma|_{(D^2)^\perp}$ of σ on $(D^2)^\perp$ are not trivial. Moreover for the points of $(D^2)^\perp\backslash(D^3)^\perp$ these kernels are one-dimensional. They form the *characteristic line distribution* of the distribution D on $(D^2)^\perp\backslash(D^3)^\perp$. The integral curves of this line distribution define a *characteristic 1-foliation* of D on $(D^2)^\perp\backslash(D^3)^\perp$. In Control Theory the leaves of this 1-foliation are also called *regular abnormal extremals of D*.

Now let us define the same objects on the projectivization of $(D^2)^\perp\backslash(D^3)^\perp$. First note that in the considered case $(D^3)^\perp$ coincides with the zero section of T^*M. So the projectivization of $(D^2)^\perp\backslash(D^3)^\perp$ is just $\mathbb{P}(D^2)^\perp$. Further note that the homotheties of the fibers of $(D^2)^\perp$ preserve the characteristic line distribution. Therefore the projectivization induces on $\mathbb{P}(D^2)^\perp$ in a natural way the *characteristic line distribution*, and the *characteristic* 1*-foliation*, which will be denoted by by $\mathcal{C}$ and $\bar{\mathcal{C}}$ respectively. The leaves of the 1-foliation $\bar{\mathcal{C}}$ will be called the *abnormal extremals of the distribution D* (on $\mathbb{P}(D^2)^\perp$). The distribution $\mathcal{C}$ can be defined equivalently in the following way: take the corank 1 distribution on $(D^2)^\perp\backslash(D^3)^\perp$, given by the Pfaffian equation $\varsigma|_{(D^2)^\perp} = 0$ and push forward it under projectivization to $\mathbb{P}(D^2)^\perp$. In this way we obtain a corank 1 distribution on $\mathbb{P}(D^2)^\perp$, which will be denoted by Δ. The distribution Δ defines the quasi-contact structure on the even dimensional manifold $\mathbb{P}(D^2)^\perp$ and $\mathcal{C}$ is exactly the characteristic distribution of this quasi-contact structure.

For the coordinate-free construction it is more convenient to work with the projectivization $\mathbb{P}(D^2)^\perp$ rather than with $(D^2)^\perp$. For simplicity, the canonical projection from $\mathbb{P}(D^2)^\perp$ to M will be denoted by π, as in the case of the cotangent bundle T^*M. Given a segment γ of abnormal extremal on $\mathbb{P}(D^2)^\perp$ one can construct two special (unparameterized) curves in three and two-dimensional projective spaces respectively. For this for any $\lambda \in \mathbb{P}(D^2)^\perp$ denote by V_λ the following subspace of $T_\lambda\mathbb{P}(D^2)^\perp$

$$V_\lambda = T_\lambda(\mathbb{P}T^*_{\pi(\lambda)}M) \cap T_\lambda\mathbb{P}(D^2)^\perp. \tag{1.2}$$

In other words, V_λ is the tangent space to the fiber of $\mathbb{P}(D^2)^\perp$ at the point λ. Note that by constructions $\dim V_\lambda = 1$. Let O_γ be a neighborhood of γ in $\mathbb{P}(D^2)^\perp$ such that $N = O_\gamma/(\bar{\mathcal{C}}|_{O_\gamma})$ is a well-defined smooth manifold. Let $\phi : O_\gamma \to N$ be the canonical projection on the factor.

First note that $\widetilde{\Delta} \stackrel{def}{=} \phi_*\Delta$ is well defined corank 1 distribution on N. Also the distribution $\widetilde{\Delta}$ is contact distribution. Second let

$$\mathcal{E}_\gamma(\lambda) = \phi_*\big(V_\lambda\big), \quad \forall \lambda \in \gamma. \tag{1.3}$$

The curve $\{\mathcal{E}_\gamma(\lambda), \lambda \in \gamma\}$ is a curve in the three-dimensional projective space $\mathbb{P}\widetilde{\Delta}(\gamma)$.

Remark 1.1. The curve E_γ is closely related to the so-called Jacobi curve J_γ of the abnormal extremal γ,

$$J_\gamma(\lambda) = \phi_*\Big(\{v \in T_\lambda\mathbb{P}(D^2)^\perp : \pi_* v \in D(\pi(\lambda))\}\Big), \tag{1.4}$$

which was intensively used in our previous papers [1], [10],[11], and [7] for construction of differential invariants and canonical frames of rank 2 distributions. The curve J_γ is the curve of 2-dimensional subspaces in the 4-dimensional space $\widetilde{\Delta}(\gamma)$. Besides, since the distribution $\widetilde{\Delta}$ is contact, the space $\widetilde{D}(\gamma)$ is endowed with the symplectic structure, defined up to a constant factor. The curve J_γ is the curve of Lagrangian subspaces w.r.t. this symplectic form. Finally, the relation between the curves J_γ and $\mathcal{E}_\gamma$ is given by the formula

$$J_\gamma(\lambda) = \pi_*\big([C,V]_\lambda\}\big) = \mathrm{span}\big\{\mathcal{E}_\gamma(\lambda), \frac{d}{dt}\mathcal{E}_\gamma\big(\Gamma(t)\big)|_{t=0}\big\}, \tag{1.5}$$

where $[C,V]_\lambda = \{[h,v](\lambda) : h \in \mathcal{C}, v \in V$ are vector fields$\}$ and $\Gamma(\cdot)$ is a parameterization of γ such that $\Gamma(0) = 0$. So, by (1.5) the Jacobi curve can be uniquely recovered from the curve $\mathcal{E}_\gamma$. $\square$

Further fix some point $\lambda \in \gamma$ and denote by $\Pi_\lambda : \widetilde{\Delta}(\gamma) \mapsto \widetilde{\Delta}(\gamma)/\mathcal{E}_\gamma(\lambda)$ the canonical projection to the factor-space. Let $L_\lambda(\bar\lambda)$=$\Pi_\lambda\big(\mathcal{E}_\gamma(\bar\lambda)\big)$ for any $\bar\lambda \in \gamma\backslash\{\lambda\}$. It is easy to show (see formula (2.4) below) that one can define $L_\lambda(\bar\lambda)$ also for $\bar\lambda = \lambda$ such that for some neighborhood I of λ on γ the curve $\{L_\lambda(\bar\lambda), \bar\lambda \in I\}$ will be a smooth curve in the projective plane $\mathbb{P}\Big(\ \bar\Delta(\gamma)/\mathcal{E}_\gamma(\lambda)\Big)$. Moreover, it is easy to see (using, for instance, Remark 1.1 above and Propositions 3.5 from [10]) that this curve has no inflection points (or, equivalently, it is regular curve in the terminology of section 2 below). In general, let Y be a three-dimensional linear space and ξ be a smooth curve without inflection points in its projectivization. It is well known (see, for example, [5], p. 53 there) that for any point $y \in \xi$ there exists the unique quadric in $\mathbb{P}Y$, which has contact of order four with ξ at y, i.e. it has the same forth jet at y as ξ. This quadric is called the *osculating quadric of* ξ *at* y. The two-dimensional quadratic cone in Y, corresponding to this quadric, is called the *osculating cone of* ξ *at* y.

By construction, there is the natural identification:

$$T_\gamma N \sim T_\lambda\mathbb{P}(D^2)^\perp/\mathcal{C}_\lambda, \tag{1.6}$$

which in turn implies that $T_\gamma N/\mathcal{E}_\gamma(\lambda)$ is naturally identified with $T_\lambda\mathbb{P}(D^2)^\perp/\mathrm{span}\{\mathcal{C}_\lambda, V_\lambda\}$. On the other hand, since V_λ is the tangent space at λ to the fiber of $\mathbb{P}(D^2)^\perp$, the latter space is naturally identified with $T_{\pi(\lambda)}M/\pi_*\mathcal{C}_\lambda$. So, finally one has the following natural identification

$$T_\gamma N/\mathcal{E}_\gamma(\lambda) \sim T_{\pi(\lambda)}M/\pi_*\mathcal{C}_\lambda. \tag{1.7}$$

Then under the identification (1.7) one has

$$\tilde\Delta(\gamma)/\mathcal{E}_\gamma(\lambda) \sim \lambda^\perp/\pi_*\mathcal{C}_\lambda, \tag{1.8}$$

where $\lambda^\perp = \{v \in T_qM : \langle\lambda, v\rangle = 0\}$.

Now fix some point $q \in M$. For any subspace W of T_qM denote by $\mathrm{pr}_W : T_qM \mapsto T_qM/W$ the canonical projection and by $\mathbb{P}(D^2)^\perp(q)$ the fiber of $\mathbb{P}(D^2)^\perp$ over q. For any $\lambda \in \mathbb{P}(D^2)^\perp(q)$ take the osculating cone of the curve $\{L_\lambda(\bar\lambda) : \bar\lambda \in \gamma_\lambda\}$ at the point λ, where γ_λ is the abnormal extremal of D, passing through λ. Under the identification (1.8) this osculating cone is the quadratic cone in the three-dimensional linear space $\lambda^\perp/\pi_*\mathcal{C}_\lambda$. Let $\mathrm{Con}(\lambda)$ be the preimage of this cone under $\mathrm{pr}_{\pi_*\mathcal{C}_\lambda}$. The set $\mathrm{Con}(\lambda)$ is the degenerated 3-dimensional quadratic cone

in the four-dimensional linear subspace $\lambda^\perp$ of T_qM. Finally, let Ξ_q be the union of the cones $\mathrm{Con}(\lambda)$ for all λ from the fiber $\mathbb{P}(D^2)^\perp(q)$,

$$\Xi_q = \bigcup_{\lambda\in\mathbb{P}(D^2)^\perp(q)} \mathrm{Con}(\lambda). \tag{1.9}$$

The set Ξ_q is a four-dimensional cone in T_qM, associated intrinsically with our distribution. What is the structure of this set? First of all it is easy to see that Ξ_q is an algebraic variety. Indeed, in the appropriate coordinates the vector fields, which span the line distributions $\mathcal{C}$ and V_λ, have the components, which are polynomials on each fiber of $\mathbb{P}(D^2)^\perp$. Therefore for any natural k the mapping from a fiber $\mathbb{P}(D^2)^\perp(q)$ to the space of k-jets of the curves in the projective plane, which assigns to any $\lambda \in \mathbb{P}(D^2)^\perp(q)$ the k-jet of the curve $\{L_\lambda(\bar\lambda) : \bar\lambda \in \gamma\}$, is a rational mapping. It implies without difficulties that the quadratic cones $\mathrm{Con}(\lambda)$ are zero levels of quadratic forms with coefficient, which are polynomials on the fiber $\mathbb{P}(D^2)^\perp(q)$. Hence Ξ_q is an algebraic variety. The main results of the present paper are the following two theorems

Theorem 1. *The set Ξ_q is a quadratic cone of signature $(3,2)$ for any $q \in M$.*

Theorem 2. *The conformal structure, defined by the field of cones $\{\Xi_q\}_{q\in M}$, coincides with the Nurowski conformal structure.*

Theorem 1 and 2 are proved in sections 2 and 4 respectively. In section 3, as a preparation to the proof of Theorem 2, we obtain the explicit formula for the constructed conformal structure in terms of the structural functions of any frame naturally adapted to the distribution. This formula is useful by itself.

1.3. **The Cartan form of (2,5)-distributions and the Wylczynski invariant of curves in the projective plane.** Both the discovery of the canonical structure for $(2,5)$-distribution in [8] and the variational approach to the equivalence problem of distributions via abnormal extremals and Jacobi curves, developed in [1] and [10], gave the new geometric interpretations of the fundamental invariant of $(2,5)$-distribution, the Cartan covariant binary biquadratic form (an invariant homogeneous polynomial of degree 4 on each plane $D(q)$). On one hand, it can be obtained from the Weyl tensor of the canonical conformal structure. On the other hand, it can be constructed from the special degree four differential on the Jacobi curves, which are curves in Lagrange Grassmannians ([11]). Using the constructions above, one can obtain one more, totally elementary interpretation of the Cartan form of a $(2,5)$-distribution in terms of the classical Wilczynski invariant ([9]) of the curves $\{L_\lambda(\bar\lambda) : \bar\lambda \in \gamma_\lambda\}$, which are curves in the projective plane. Let us briefly describe how to do it, referring to the Appendix for the facts from the theory of curves in the projective spaces. Actually by our method we can construct more than just quadratic cone in the tangent space T_qM of any point q. For this take the union of the preimages of the curves $\{L_\lambda(\bar\lambda) : \bar\lambda \in \gamma_\lambda\}$ under the projection $\mathrm{pr}_{\pi_*\mathcal{C}_\lambda}$, where λ runs over the fiber $\mathbb{P}(D^2)^\perp(q)$, namely, let

$$\Omega_q = \bigcup_{\lambda\in\mathbb{P}(D^2)^\perp(q)} \mathrm{pr}^{-1}_{\pi_*\mathcal{C}_\lambda}\Big(\{L_\lambda(\bar\lambda) : \bar\lambda \in \gamma_\lambda\}\Big). \tag{1.10}$$

The set Ω_q is a conic hypersurface in T_qM. In order to get the value of the Cartan form at some vector $v \in D(q)$ one can proceed as follows: First note that there exists exactly one point λ on the fiber of $\mathbb{P}(D^2)^\perp$ over the point q such that the projection to the base manifold M of the abnormal extremal γ_λ, passing through λ, is tangent to v. From (1.10) it follows that the intersection of the set Ω_q with the hyperplane $\lambda^\perp$ is the hypersurface in $\lambda^\perp$, which is exactly the preimage of the curve $\{L_\lambda(\bar\lambda) : \bar\lambda \in \gamma_\lambda\}$ under pr_v. Further recall that for any

regular curve in the projective plane one can construct the invariant degree 3 differential, i.e. a degree 3 homogeneous polynomial on the tangent line to the curve at any point (the (first) Wilczinsky invariant in the terminology of the Appendix below). Applying Proposition 5.1 from the Appendix to the curve $\{L_\lambda(\bar\lambda) : \bar\lambda \in \gamma_\lambda\}$ we get that for this curve this degree 3 differential is equal to zero at the point λ. Therefore the derivative of this differential at λ is well-defined degree 4 homogeneous polynomial on the tangent space to the curve $\{L_\lambda(\bar\lambda) : \bar\lambda \in \gamma_\lambda\}$ at λ. Take a parameter t on γ_λ such that $\gamma_\lambda(0) = \lambda$ and $\pi_*\dot\gamma_\lambda(0) = v$. It turns out the value of the Cartan form at v is equal, up to the constant factor $-\frac{1}{5}$, to the value of the mention degree 4 homogeneous polynomial at $\frac{d}{dt}L_\lambda(\gamma_\lambda(t))|_{t=0}$. The last statement can be obtained immediately if one combines Remark 1.1, Proposition 5.1 and Remark 5.1 from the Appendix with Theorem 2 from [11]. It also shows that the fundamental invariant of $(2,5)$-distribution at a point q can be obtained as an invariant of the germ of the set Ω_q at 0.

1.4. **Other types of distributions.** Now we say several words about similar constructions for other types of distributions. First note that shortly after Nurowski's discovery, R. Bryant gave a construction of the canonical conformal structure of signature $(3,3)$ for maximally nonholonomic $(3,6)$-distributions ([3]), exploiting again the Cartan method of equivalence. What are the analogs of the Nurowski and Bryant conformal structures for other classes of distributions? It turns out that the method of osculating cones in the projective plane given in the present paper is a particular case of rather general procedure, which allows to assign natural algebra-geometric structures to vector distributions from very wide class. Let us describe this procedure very briefly. First, similarly to above one can define the 1-foliation of abnormal extremals of a distribution D in some submanifold $\mathcal{S}$ of the cotangent bundle, where $\mathcal{S} = D^\perp$ in the case of odd rank and $\mathcal{S}$ is a codimension one subbundle of $D^\perp$ with algebraic fibers in the case of even rank (in the case of rank 2 distributions $\mathcal{S} = (D^2)^\perp$)[1] . Second, studying the dynamics of the fibers of $\mathcal{S}$ along the foliation of abnormal extremals, for any $q \in M$ one gets an algebraic variety $\mathfrak{K}_q$ on the linear space $T_qM \oplus D^\perp(q)$. In the case of maximally nonholonomic $(2,5)$ and $(3,6)$ distributions the projection of this variety to T_qM gives the quadratic cone there, which defines the Nurowski and the Bryant conformal structure respectively. Moreover, the variety $\mathfrak{K}_q$ can be reconstructed from this projection. In general case, the variety $\mathfrak{K}_q$ can not be recovered from its projection to T_qM (for example, in many cases this projection is the whole T_qM). However the constructed varieties $\mathfrak{K}_q$, $q \in M$ in the vector bundle $TM \oplus D^\perp$ contain the fundamental information about the distribution. The details of these constructions will be given in our forthcoming paper [2].

2. Proof of Theorem 1

2.1. **Preliminaries.** We start with several simple auxiliary facts from geometry of curves in projective spaces. Let Z be an k-dimensional linear space and ζ be a smooth curve in the projective space $\mathbb{P}Z$ or, equivalently, a smooth curve of straight lines in Z. Take some parameter $t \in \tilde I$ on ζ, where $\tilde I$ is some segment in $\mathbb{R}$. The curve $\varepsilon : \tilde I \mapsto Z$ such that $\varepsilon(t) \in \xi(t)$ for all $t \in \tilde I$ will be called a *representative of the curve ξ, corresponding to the parameter t*. A smooth curve ζ in the projective space $\mathbb{P}Z$ is called *regular*, if some (and therefore any) its representative $\epsilon : \tilde I \mapsto Z$ satisfies

$$\operatorname{span}\{\varepsilon^{(i)}(t)\}_{i=0}^{k-1} = Z, \quad \forall t \in \tilde I, \tag{2.1}$$

Assume that ζ is a regular curve and t is some parameter on it. It is well-known that there exists the unique, up to the multiplication on a constant, representative ε, corresponding to

[1] If rank of a distribution is even and not 2, then the abnormal extremals may not exist. However the procedure can be extended also to this case after the complexification of the fibers of T^*M.

the parameter t such that

$$\varepsilon^{(k)}(t) = \sum_{i=0}^{k-2} B_i(t)\varepsilon^{(i)}(t), \tag{2.2}$$

i.e. the coefficients near $\varepsilon^{(k-1)}(t)$ in the linear decomposition of $\varepsilon^{(k)}(t)$ w.r.t. the basis (2.1) vanishes. Such representative of ξ will be called *canonical w.r.t. the parameter t*.

Remark 2.1. If in addition $\dim Z = k$ is even, $k = 2m$, the space Z is endowed with the symplectic form $\bar{\sigma}$, and the m-dimensional subspace $\operatorname{span}\{\varepsilon^{(i)}(t)\}_{i=0}^{m-1}$ is Lagrangian for any $t \in \tilde{I}$, then the representative $t \mapsto \varepsilon(t)$ is canonical if and only if

$$\bar{\sigma}\big(\varepsilon^{(m)}(t), \varepsilon^{(m-1)}(t)\big) \equiv \text{const}. \tag{2.3}$$

Also in this case $B_{2m-3}(t) = B'_{2m-2}(t)$ and more general for any $0 \le j \le m-2$ the coefficient B_{2j-1} is equal to some polynomial expression w.r.t. the tuple of the coefficients $\{B_{2l}\}_{l=j}^{m}$ and their derivatives. All statements in this remark can be easily checked (see also [12]). □

Further, fix some point $\ell \in \zeta$, $\ell = \zeta(t_0)$. Note that ℓ is a straight line in Z. Let $\Pi_\ell : Z \mapsto Z/\ell$ be the canonical projection to the factor-space. If ε is a representative of ζ, corresponding to the parameter t, then the germ at t_0 of the following map

$$\tilde{\varepsilon}(t) = \begin{cases} \Pi_\ell\big(\varepsilon(t)\big) & t \neq t_0 \\ \Pi_\ell\big(\varepsilon'(t_0)\big) & t = t_0 \end{cases} \tag{2.4}$$

is the representative of the smooth curve in $(k-2)$-dimensional projective space $\mathbb{P}(Z/\ell)$. This curve will be called the *reduction of the curve ζ by the point ℓ*. The point $y = \mathbb{R}\Pi_\ell\big(\varepsilon'(t_0)\big)$ will be called the *point of the reduction, corresponding to ℓ*.

Remark 2.2. Note also that if the curve ζ is regular then its reduction by ℓ is regular in a neighborhood of the point of the reduction, corresponding to ℓ. □

Now let Y be a three-dimensional linear space, ξ be a regular curve in the projective plane $\mathbb{P}Y$, and y be a point on ξ. In the following lemma we give an explicit formula for the osculating quadric to ξ at y in some coordinates under normalization assumptions:

Lemma 2.1. *Suppose that (y_1, y_2) is a coordinate system in $\mathbb{P}Y$ such that $y = (0,0)$ and $\tilde{\varepsilon}$ is a representative of ξ such that $\tilde{\varepsilon}(0) = y$, $\tilde{\varepsilon}(t) = \big(\alpha_1(t), \alpha_2(t)\big)$ in the chosen coordinates, and*

$$\dot{\alpha}_1(0) = \frac{1}{2},\ \dot{\alpha}_2(0) = \ddot{\alpha}_1(0) = 0, \ddot{\alpha}_2(0) = \frac{1}{3}. \tag{2.5}$$

Then the osculating quadric to ξ at y has the following equation

$$\frac{2}{3}y_1^2 + 2\alpha_2^{(3)}(0)y_1y_2 - \left(4\alpha_1^{(3)}(0) + 6\big(\alpha_2^{(3)}(0)\big)^2 - \frac{3}{2}\alpha_2^{(4)}(0)\right)y_2^2 - y_2 = 0. \tag{2.6}$$

The proof of Lemma 2.1 consists of straightforward computations. The normalization assumptions (2.5) are convenient when one makes a reduction. Indeed, suppose that $\dim Z = 4$, $Y = Z/\ell$ and the curve ξ is the reduction of the curve ζ by a point ℓ. Let ε be a representative of ζ such that $\varepsilon(0) = \ell$, $\tilde{\varepsilon}$ be the representative of ξ, satisfying (2.4) and $y = \tilde{\varepsilon}(0)$. As a basis in Y take the tuple $\Big(\Pi\big(\varepsilon'(0)\big), \Pi\big(\varepsilon''(0)\big), \Pi\big(\varepsilon^{(3)}(0)\big)\Big)$. Let (Y_0, Y_1, Y_2) be the coordinates in Y w.r.t. this basis. Then $(y_1, y_2) = (Y_1/Y_0, Y_2/Y_0)$ are coordinates in a neighborhood of y in $\mathbb{P}Y$. If $\tilde{\varepsilon}(t) = \big(\alpha_1(t), \alpha_2(t)\big)$ in this coordinates, then by direct check

the functions α_1 and α_2 satisfy (2.5). Besides, if ε is the canonical representative of the curve ζ, satisfying (2.2), where $k=4$, then by simple computations

$$\alpha_1^{(3)}(0)=\frac{B_2(0)}{4},\quad \alpha_2^{(3)}(0)=0,\quad \alpha_2^{(4)}(0)=\frac{B_2(0)}{5}.$$

Then from Lemma 2.1 it follows that the osculating cone of ξ at y in Y w.r.t. the chosen coordinates (Y_0,Y_1,Y_2) has the following equation

$$\frac{2}{3}Y_1^2-\frac{7}{10}B_2(0)Y_2^2-Y_0Y_2=0. \tag{2.7}$$

Remark 2.3. Comparing decomposition (2.2) and formula (4.2) from [10], we get that the function B_2 coincides, up to a constant factor, with the so-called Ricci curvature ρ of the parameterized curve $t\mapsto \operatorname{span}\{\epsilon(t),\epsilon'(t)\}$ in the Grassmannian $G_2(Z)$ of half-dimensional subspaces of the space Z. More precisely, $B_2=-\frac{15}{2}\rho$. The definition of the Ricci curvature of the curves in Grassmannians of half-dimensional subspaces of an even-dimensional space can be found, for example, in subsection 2.3 of [10]. $\square$

2.2. **Algebraic structure of the set Ξ_q.** Note that the curve $\mathcal{E}_\gamma$, constructed in the subsection 1.2 for a segment γ of any abnormal extremal, is a regular curve in the corresponding three-dimensional projective space (this fact follows, for example, from Remark 1.1 above and Proposition 3.5 from [10]). The curve L_λ, defined in the subsection 1.2 as well, is exactly the reduction of $\mathcal{E}_\gamma$ by $\mathcal{E}_\gamma(\lambda)$. Hence, by Remark 2.2 the curve L_λ is a regular curve in the corresponding projective plane. Now we will use Lemma 2.1 for the explicit calculation of the set Ξ_q in appropriate coordinates. For this it is more convenient to work with $(D^2)^\perp$ rather then with its projectivization. To begin with given two vector fields X_1, X_2, constituting a local basis of distribution D, one can construct a special vector field $\vec{h}_{X_1,X_2}$ tangent to the characteristic 1-foliation of D on $(D^2)^\perp\backslash(D^3)^\perp$. For this suppose that

$$\begin{aligned} &X_3=[X_1,X_2]\mod D,\ \ X_4=[X_1,[X_1,X_2]]=[X_1,X_3]\mod D^2,\\ &X_5=[X_2,[X_1,X_2]]=[X_2,X_3]\mod D^2 \end{aligned} \tag{2.8}$$

The tuple $\{X_i\}_{i=1}^5$ is called the *adapted frame of the distribution* D. Let us introduce "quasi-impulses" $u_i:T^*M\mapsto\mathbb{R}$, $1\le i\le 5$, by the following formula: $u_i(\tilde\lambda)=\langle\tilde\lambda,X_i(q)\rangle$. For given function $G:T^*M\mapsto\mathbb{R}$ denote by $\vec{G}$ the corresponding Hamiltonian vector field defined by the relation $\sigma(\vec{G},\cdot)=-d\,G(\cdot)$. Then it is easy to show (see, for example [13]) that the following vector field

$$\vec{h}_{X_1,X_2}=u_4\vec{u}_2-u_5\vec{u}_1 \tag{2.9}$$

is tangent to the characteristic 1-foliation. In the sequel we will work with the fixed local basis (X_1,X_2), therefore we will write $\vec{h}$ instead of $\vec{h}_{X_1,X_2}$.

Further denote by $\mathcal{P}:T^*M\mapsto\mathbb{P}T^*M$ the canonical projection. Let $\vec{e}$ be the Euler field, i.e., the infinitesimal generator of homotheties on the fibers of T^*M. Take a point $\tilde\lambda\in(D^2)^\perp\backslash(D^3)^\perp$ and let $\tilde\gamma$ be a segment of the abnormal extremal of D in $(D^2)^\perp\backslash(D^3)^\perp$, passing through $\tilde\lambda$. Let $\lambda=\mathcal{P}(\tilde\lambda)$ and $\gamma=\mathcal{P}(\tilde\gamma)$. Then, combining the identification (1.6) and the identification $T_\lambda\mathbb{P}(D^2)^\perp\sim T_{\tilde\lambda}(D^2)^\perp/\mathbb{R}\vec{e}(\tilde\lambda)$, we get

$$T_\gamma N\sim T_{\tilde\lambda}(D^2)^\perp/\operatorname{span}\{\vec{h}(\tilde\lambda),\vec{e}(\tilde\lambda)\}. \tag{2.10}$$

By analogy with (1.2) let $\widetilde{V}_{\tilde\lambda}$ be the tangent to the fiber of $(D^2)^\perp$ at $\bar\lambda\in(D^2)^\perp\backslash(D^3)^\perp$, namely,

$$\widetilde{V}_{\tilde\lambda}=T_{\tilde\lambda}(T^*_{\pi(\tilde\lambda)}M)\cap T_{\tilde\lambda}(D^2)^\perp. \tag{2.11}$$

Then it is not hard to see that under identification (2.10) one has

$$\mathcal{E}_\gamma\big(\mathcal{P}(e^{tH}\tilde\lambda)\big) = (e^{-t\vec h})_*\big(\widetilde V(e^{t\vec h}\tilde\lambda)\big)/\mathrm{span}\{\vec h(\tilde\lambda), \vec e(\tilde\lambda)\} \tag{2.12}$$

where $e^{t\vec h}$ is the flow generated by the vector field $\vec h$. The pair of functions (u_4, u_5) defines the coordinates on any fiber of $(D^2)^\perp$. Consider the following vector field on $(D^2)^\perp\backslash(D^3)^\perp$, tangent to the fibers of $(D^2)^\perp$:

$$\epsilon_1(\bar\lambda) = \gamma_4(\bar\lambda)\partial_{u_4} + \gamma_5(\bar\lambda)\partial_{u_5}, \tag{2.13}$$

where

$$\gamma_4(\bar\lambda)u_5 - \gamma_5(\bar\lambda)u_4 \equiv \mathrm{const}. \tag{2.14}$$

Let

$$v_{\tilde\lambda}(t) = (e^{-t\vec h})_*\epsilon_1\big(e^{t\vec h}(\tilde\lambda)\big). \tag{2.15}$$

Without introducing additional notations, we will look on $v_{\tilde\lambda}(t)$ both as the element of $T_{\tilde\lambda}(D^2)^\perp$ and the element of the factor-space $T_{\tilde\lambda}(D^2)^\perp/\mathrm{span}\{\vec h(\tilde\lambda), \vec e(\tilde\lambda)\}$, i.e., the image of $v_{\tilde\lambda}(t)$ under this factorization. Combining Remarks 1.1, 2.1, and the proof of Proposition 4.4 in [10], we obtain that under the identification (2.12) the curve $t \mapsto v_{\tilde\lambda}(t)$ is the canonical representative of the curve $\mathcal{E}_\gamma$ w.r.t. the parameterization $t \mapsto \mathcal{E}_\gamma\big(\mathcal{P}(e^{tH}\tilde\lambda)\big)$. Note that from (2.15) and the well known property of Lie brackets it follows that

$$v^{(i)}_{\tilde\lambda}(0) = (\mathrm{ad}\,\vec h)^i\epsilon_1(\tilde\lambda) \tag{2.16}$$

(here, as usual, $(\mathrm{ad})\,\vec h\mathcal{X} = [\vec h, \mathcal{X}]$ for a vector field $\mathcal{X}$). From (2.16) and the definition of the canonical representative (see (2.2)) it follows that there exists the tuple of functions $\{\mathcal{B}_i(\tilde\lambda)\}_{i=0}^2$ on $(D^2)^\perp\backslash(D^3)^\perp$ such that

$$(\mathrm{ad}\,\vec h)^4\epsilon_1(\tilde\lambda) = \sum_{i=0}^{2}\mathcal{B}_i(\tilde\lambda)(\mathrm{ad}\,\vec h)^i\epsilon_1(\tilde\lambda) \tag{2.17}$$

(compare also with formula (4.10) in [10]). From the last formula and (2.7) it follows that the set $\mathrm{Con}(\mathcal{P}\tilde\lambda)$ defined in subsection 1.2 has the following description: If (s, Y_0, Y_1, Y_2) are the coordinates in $(\mathcal{P}\tilde\lambda)^\perp$ w.r.t. the basis

$$\Big(\pi_*\big(\vec h(\tilde\lambda)\big), \pi_*\big([\vec h, \epsilon_1](\tilde\lambda)\big), \pi_*\Big((\mathrm{ad}\,\vec h)^2\epsilon_1(\tilde\lambda)\Big), \pi_*\Big((\mathrm{ad}\,\vec h)^3\epsilon_1(\tilde\lambda)\Big)\Big), \tag{2.18}$$

then the set $\mathrm{Con}(\mathcal{P}\tilde\lambda)$ has the following equation in these coordinates:

$$\frac{2}{3}Y_1^2 - \frac{7}{10}\mathcal{B}_2(\tilde\lambda)Y_2^2 - Y_0Y_2 = 0. \tag{2.19}$$

For simplicity take $\gamma_4 = \frac{1}{u_5}$ and $\gamma_5 \equiv 0$. Then by direct calculations (one can use also formulas (4.24), (4.35) and Lemmas 4.2 and 4.3 of [11]) it is not difficult to show that

$$\begin{aligned}
\pi_*\big([\vec h, \epsilon_1]\big) &= -\frac{1}{u_5}X_2 \quad \mathrm{mod}\{\mathbb{R}\pi_*\vec h\},\\
\pi_*\big((\mathrm{ad}\,\vec h)^2\epsilon_1\big) &= \frac{l_1}{u_5}X_2 + X_3 \quad \mathrm{mod}\{\mathbb{R}\pi_*\vec h\},\\
\pi_*\big((\mathrm{ad}\,\vec h)^3\epsilon_1\big) &= \frac{Q}{u_5}X_2 + l_2X_3 + u_4X_5 - u_5X_4 \quad \mathrm{mod}\{\mathbb{R}\pi_*\vec h\},
\end{aligned} \tag{2.20}$$

where the functions l_1 and l_2 are linear and the function Q is a quadratic form on each fiber of $(D^2)^\perp$ (in section 3 below we give the explicit formulas for this functions in terms of the structural functions of an adapted frame $\{X_i\}_{i=1}^5$, see formulas (3.5)-(3.7)). Besides, from

the proof of Theorem 3 in [10] it follows that the function $\mathcal{B}_2$, appearing in the equation (2.19), is a quadratic form on each fiber of $(D^2)^\perp(q)$ (see also explicit formula (3.8) in section 3 below).

Now fixing the point $q \in M$, we will look at the functions l_i, Q, and $\mathcal{B}_2$ as at the functions of (u_4, u_5) (which constitute coordinates on the fiber $(D^2)^\perp(q)$) and we will write $l_i(u_4, u_5)$, $Q(u_4, u_5)$, and $\mathcal{B}_2(u_4, u_5)$ correspondingly. Then, running over the fiber $(D^2)^\perp(q)$ and using (2.19) and (2.20), one can obtain by straightforward calculations that if $(x_1, \dots, x_5)$ are coordinates in T_qM w.r.t. the basis $\big(X_1(q), \dots, X_5(q)\big)$ then the set Ξ_q, defined in (1.9), has the following equation in these coordinates:

$$\begin{aligned} &x_1x_5 - x_2x_4 + \frac{2}{3}\Big(x_3 - l_2(x_5, -x_4) - \frac{3}{4}l_1(x_5, -x_4)\Big)^2 - \\ &\Big(\frac{7}{10}\mathcal{B}_2(x_5, -x_4) + Q(x_5, -x_4) + \frac{3}{8}\big(l_1(x_5, -x_4)\big)^2\Big) = 0. \end{aligned} \tag{2.21}$$

Taking into account the algebraic structure of the functions l_1, Q, and $\mathcal{B}_2$, we get that Ξ_q is a quadratic cone. Besides, it is easy to see that the quadratic form in the left handside of (2.21) has signature $(3, 2)$. The proof of Theorem 1 is completed.

The conformal structure, defined by the field of cones $\{\Xi_q\}_{q\in M}$ will be called the *canonical conformal structure, associated with the distribution D*.

3. The canonical conformal structure in terms of the structural functions of an adapted frame

Let, as before, $\{X_i\}_{i=1}^5$, be an adapted frame of the $(2,5)$-distribution D and u_i, $1 \le i \le 5$, be the corresponding quasi-impulses. Denote by c_{ji}^k the structural functions of the frame $\{X_i\}_{i=1}^5$, i.e., the functions, satisfying

$$[X_i, X_j] = \sum_{k=1}^5 c_{ji}^k X_k.$$

We are going to express the quadratic cone Ξ_q, $q \in M$, in terms of the structural functions c_{ji}^k. For this we slightly modify the computations produced in section 4 of [11]. As in [11], denote

$$b(u_4, u_5) = \frac{1}{3}\Big((c_{42}^4 + c_{52}^5)u_4 - (c_{41}^4 + c_{51}^5)u_5\Big), \tag{3.1}$$

$$b_1(u_4, u_5) = c_{32}^3 u_4 - c_{31}^3 u_5, \tag{3.2}$$

$$\alpha_3(u_4, u_5) = c_{52}^3 u_4^2 - (c_{42}^3 + c_{51}^3)u_4u_5 + c_{41}^3 u_5^2, \tag{3.3}$$

$$\Pi(u_4, u_5) = (c_{32}^1 + c_{53}^4)u_4^2 + (c_{32}^2 - c_{31}^1 - c_{43}^4 + c_{53}^5)u_4u_5 - (c_{31}^2 + c_{43}^5)u_5^2. \tag{3.4}$$

Then from formulas (4.3) and (4.36) of [11] it is not hard to get that the function l_1, introduced in (2.20), satisfies

$$l_1 = b_1 + 3b \tag{3.5}$$

(note that ϵ_1 here is the same as $6\epsilon_1$ in [11]). From formulas (4.3), (4.31)-(4.37), and (4.44) of [11] one can obtain by direct computations that the functions l_2 and Q from (2.20) satisfy:

$$l_2 = -3b, \tag{3.6}$$

$$Q = 6\vec{h}(b) + \vec{h}(b_1) + \Pi - \alpha_3 - b_1^2 - 3bb_1 - 9b^2. \tag{3.7}$$

Besides, combining Remark 2.3 with formula (4.56) from [11], we have

$$\mathcal{B}_2 = 2\alpha_3 - \Pi - \vec{h}(b_1) - 9\vec{h}(b) + b_1^2 + 9b^2. \tag{3.8}$$

Finally, substituting expressions (3.5)-(3.8) into (2.21) we get easily that if $(x_1, \dots, x_5)$ are coordinates in T_qM w.r.t. the basis $\big(X_1(q), \dots, X_5(q)\big)$ then the cone Ξ_q has the following equation in these coordinates

$$\begin{aligned} &x_1x_5 - x_2x_4 + \frac{2}{3}\Big(x_3 - \frac{3}{4}(b_1 - b)(x_5, -x_4)\Big)^2 - \\ &\frac{3}{10}\Big(\Pi + \frac{4}{3}\alpha_3 + \vec{h}(b_1 - b) + \frac{1}{4}(b_1 - b)(b_1 + 9b)\Big)(x_5, -x_4) = 0. \end{aligned} \tag{3.9}$$

In the last formula for shortness, instead of writing the arguments $(x_5, -x_4)$ after each function b, b_1, α_3 and Π, we write them after the polynomial expressions, involving these functions. So, for example, $(b_1 - b)(x_5, -x_4)$ means $b_1(x_5, -x_4) - b(x_5, -x_4)$. Note that from (3.1)-(3.4) and (2.9) it follows that b and b_1 are linear forms in u_4 and u_5, while α_3, Π, $\vec{h}(b_1 - b)$ are quadratic forms in u_4 and u_5. Therefore the equation (3.9) together with expressions (3.1)-(3.4) and (2.9) gives the explicit formula for the cone Ξ_q it terms of the structural functions of the adapted frame $\{X_i\}_{i=1}^5$.

4. Proof of Theorem 2

We start with the brief description of the construction of the Nurowski conformal structure, associated with the $(2, 5)$-distribution D. His construction is based strongly on the chapter VI of the E. Cartan paper [4], where the canonical coframe for this type of distributions on some 14-dimensional principal bundle over M was constructed. To begin with let $\{\omega_i\}_{i=1}^5$ be a coframe on M such that the rank 2 distribution D is the annihilator of the first three elements of this coframe, namely,

$$D(q) = \{v \in T_qM;\ \omega_1(v) = \omega_2(v) = \omega_3(v) = 0\}, \quad \forall q \in M \tag{4.1}$$

Obviously the bundle of all coframes satisfying (4.1) has 19-dimensional fibers. After quite cumbersome algebraic manipulation, E. Cartan succeed to distinguish a special 12-dimensional subbundle $\mathcal{K}(M)$ of this coframe bundle (having 7-dimensional fibers), which is characterized as follows: A coframe $\{\omega_i\}_{i=1}^5$ belongs to the coframe bundle $\mathcal{K}(M)$ if and only if it satisfies the following structural equations (formula (VI.5) in [4]):

$$\begin{aligned} d\,\omega_1 &= \omega_1 \wedge (2\overline{\omega}_1 + \overline{\omega}_4) + \omega_2 \wedge \overline{\omega}_2 + \omega_3 \wedge \omega_4 \\ d\,\omega_2 &= \omega_1 \wedge \overline{\omega}_3 + \omega_2 \wedge (\overline{\omega}_1 + 2\overline{\omega}_4) + \omega_3 \wedge \omega_5 \\ d\,\omega_3 &= \omega_1 \wedge \overline{\omega}_5 + \omega_2 \wedge \overline{\omega}_6 + \omega_3 \wedge (\overline{\omega}_1 + \overline{\omega}_4) + \omega_4 \wedge \omega_5 \\ d\,\omega_4 &= \omega_1 \wedge \overline{\omega}_7 + \frac{4}{3}\omega_3 \wedge \overline{\omega}_6 + \omega_4 \wedge \overline{\omega}_1 + \omega_5 \wedge \overline{\omega}_2 \\ d\,\omega_5 &= \omega_2 \wedge \overline{\omega}_7 - \frac{4}{3}\omega_3 \wedge \overline{\omega}_5 + \omega_4 \wedge \overline{\omega}_3 + \omega_5 \wedge \overline{\omega}_4, \end{aligned} \tag{4.2}$$

where $\overline{\omega}_j$, $1 \le j \le 7$, are new 1-forms. It turns out that a further "reduction" of the coframe bundle $\mathcal{K}_1(M)$ (i.e., a selection of an intrinsic proper subbundle of it) is impossible. Of course, it is only the first step in Cartan's construction of the canonical coframe, but this

step is already enough for our purposes. The remarkable observation of P. Nurowski was that the field of cones $\{\mathcal{N}_q\}_{q\in M}$, satisfying

$$\mathcal{N}_q = \left\{ X \in T_qM : \omega_1(X)\omega_5(X) - \omega_2(X)\omega_4(X) + \frac{2}{3}\big(\omega_3(X)\big)^2 = 0 \right\}, \tag{4.3}$$

does not depend on the choice of a coframe $\{\omega_i\}_{i=1}^5$, satisfying (4.2).

Now we have all tools to prove that our cones Ξ_q coincides with the Nurowski cones $\mathcal{N}_q$. Fix some coframe $\{\omega_i\}_{i=1}^5$ satisfying (4.2). Let $\{\widetilde{X}_k\}_{k=1}^5$, be the frame on M dual to the coframe $\{\omega_i\}_{i=1}^5$, namely, $\omega_i(\widetilde{X}_k) = \delta_{i,k}$. Denote by

$$X_k = \widetilde{X}_{6-k}, \quad 1 \le k \le 5. \tag{4.4}$$

Any frame $\{X_i\}_{i=1}^5$ obtained in such way will be called the *Cartan frame* of the distribution D. From (4.2) it is not difficult to show that any Cartan frame is adapted frame of the distribution D and to find its structural functions (see formulas (5.5)-(5.13) in [11]). Moreover, as already was noticed in [11] (formulas (5.15) and (5.16) there), for any Cartan's frame

$$b_1 = b, \quad \Pi = -\frac{4}{3}\alpha_3. \tag{4.5}$$

Substituting the last relations into (3.9) we obtain that for any Cartan frame $\{X_i\}_{i=1}^5$, if $(x_1,\dots,x_5)$ are coordinates in T_qM w.r.t. the basis $\big(X_1(q),\dots,X_5(q)\big)$ then the cone Ξ_q has the following equation in these coordinates

$$x_1x_5 - x_2x_4 + \frac{2}{3}x_3^2 = 0. \tag{4.6}$$

Comparing this equation with (4.3) and taking into account that the coframe $\{\omega_i\}_{i=1}^5$ is dual to the frame $\{X_{6-i}\}_{i=1}$, we get immediately that $\Xi_q = \mathcal{N}_q$, which completes the proof of Theorem 2.

Actually one can relate to Nurowski's equation (4.3) for the canonical cones, as to the particular case of our equation (3.9) applied to a Cartan frame.

5. Appendix

In subsection 1.3 we gave the description of the Cartan covariant binary biquadratic form of $(2,5)$-distribution in terms of the Wilczynski invariant of certain curves in the projective plane. In the present Appendix we collect several facts about the invariants of curves in the projective spaces and their reductions, on which this description was based.

First of all let us recall, what the Wilczynski invariants of curves in projective spaces ([9],[6]) are. As in subsection 2.1 , let Z be an k-dimensional linear space and ζ be a regular curve in the projective space $\mathbb{P}Z$. It turns out that the set of all parameters t on ζ, such that for their canonical representative ε the coefficient B_{k-2} in the decomposition (2.2) is identically equal to zero, defines *the* canonical projective structure on the curve ζ (i.e., any two parameters from this set are transformed one to another by Möbius transformation). It follows from the following reparameterization rule for the coefficient B_{k-2}: if τ is another parameter, $t = \varphi(\tau)$, and $\widetilde{B}_{k-2}$ is the coefficient from the decomposition (2.2) corresponding to the canonical representative of ζ w.r.t. the parameter τ, then

$$\widetilde{B}_{k-2}(\tau) = \dot{\varphi}(\tau)^2 B_{k-2}(\varphi(\tau)) - \frac{n(n^2-1)}{6}\mathbb{S}(\varphi), \tag{5.7}$$

where $\mathbb{S}(\varphi)$ is a Schwarzian derivative of φ, $\mathbb{S}(\varphi) == \frac{d}{dt}\left(\frac{\varphi''}{2\varphi'}\right) - \left(\frac{\varphi''}{2\varphi'}\right)^2$. Any parameter from the canonical projective structure of ζ is called *a projective parameter*.

Now let t be a projective parameter on ζ. E. Wilczynski showed that for any i, $1 \le i \le k-2$, the following degree $i+2$ differentials

$$\mathcal{W}_i(t) \stackrel{def}{=} \frac{(i+1)!}{(2i+2)!}\left(\sum_{j=1}^{i}(-1)^{j-1}\frac{(2i-j+3)!(k-i+j-3)!}{(i+2-j)!j!}B^{(j-1)}_{k-3-i+j}(t)\right)(dt)^{i+2} \tag{5.8}$$

on ζ does not depend on the choice of the projective parameters a The form $\mathcal{W}_i$ is called *the i-th Wilczynski invariant of the curve ζ*. In particular, the first and the second Wilczynski invariants, which will be used in the sequel, have the form

$$\mathcal{W}_1 = B_{k-3}(t)(dt)^3, \quad k \ge 3; \tag{5.9}$$

$$\mathcal{W}_2 = \left(B_{k-4}(t) - \frac{k-3}{4}B'_{k-3}(t)\right)(dt)^4, \quad k \ge 4 \tag{5.10}$$

Remark 5.1. In a continuation of Remark 2.1, if $\dim Z = k$ is even, $k = 2m$, the space Z is endowed with the symplectic form $\bar{\sigma}$, the m-dimensional subspaces $\mathrm{span}\{\varepsilon^{(i)}(t)\}_{i=0}^{m-1}$ is Lagrangian for any t, and t is a projective parameter, then $B_{2m-3} \equiv B'_{2m-2} \equiv 0$. So,

$$\mathcal{W}_1 \equiv 0, \quad \mathcal{W}_2 = B_{2m-4}(t)dt^4. \tag{5.11}$$

Besides, it can be shown that $\mathcal{W}_2$ coincides, up to some constant factor C_m, with the fundamental form of the curve of Lagrangian subspaces $\mathrm{span}\{\varepsilon^{(i)}(t)\}_{i=0}^{m-1}$, introduced in [1] (for its definition see also [10, subsection 2.3]). In the case $m = 2$, corresponding to $(2,5)$-distributions, this fact follows from formula (4.2) of [10] and the factor C_2 is equal to 35. □

Now let Y be a three-dimensional linear space, ξ is a regular curve in the projective space $\mathbb{P}Y$ and the first Wilczynski invariant $\mathcal{W}_1$ of ξ is equal to zero at some point $y \in \xi$. If t is a projective parameter, $t = t_0$ corresponds to the point y and $B_{k-3}(t)$ is the corresponding coefficient in the decomposition (2.2), then $B_{k-3}(t_0) = 0$ and the following form $\mathcal{W}_1^{(1)} = B'_{k-3}(t_0)dt^4$ is the well-defined degree 4 differential at the point y on the curve. This degree 4 differential will be called the *derivative of the first Wilczynski invariant at ℓ*. Note also that in the considered case there is only one Wilczynski invariant, the first one, so, referring to it, one usually omits the word "first". Using formulas (5.11), it is not difficult to get the following

Proposition 5.1. *Suppose that* $\dim Z = 4$, *the space Z is endowed with the symplectic form $\bar{\sigma}$, ζ is a regular curve in $\mathbb{P}Z$ such that for its representative $t \mapsto \varepsilon(t)$ all 2-dimensional subspaces* $\mathrm{span}\{\varepsilon(t), \varepsilon'(t)\}$ *are Lagrangian. If ξ is the reduction of the curve ζ by a point ℓ, then the Wilczynski invariant of ξ at the point y of the reduction, corresponding to ℓ, is equal to zero and the derivative of the Wilczynski invariant of the curve ξ at y is equal, up to the factor $\frac{1}{5}$, to the second Wilczynski invariant of the curve ζ at ℓ.*

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