Journal of Dynamical and Control Systems, Vol. 8, No. 2, April 2002, 167–215 (© 2002)

GEOMETRY OF JACOBI CURVES. II

A. AGRACHEV and I. ZELENKO

ABSTRACT. This is the second part of the work on Geometry of curves in the Lagrange Grassmannians. Part I is published in the same Journal, Vol. 8, No. 1, pp. 93–140. Here we study an important class of *flat curves* and give the estimates for the conjugate points. The estimates are presented in the form of *comparison theorems*. We use terminology and notations introduced in Part I. In order to make references to Part I we put I. before the number of the corresponding formula, section, theorem, proposition or lemma from Part I.

1. FLAT CURVES

In the present section we introduce and investigate the "straightest" curves of the given constant rank in the Lagrange Grassmannian. We call them *flat curves*. Basic symplectic invariants of a curve measure its deviation from the flat curve and thus play the role of curvatures.

First let us give a general definition of the flat curves which does not involve any specific information about the rank. Let, as before, W be the $2m$-dimensional linear symplectic space. Let $\Lambda : I \mapsto L(W)$ be an ample curve of the constant weight, defined on the interval $I \subseteq \mathbb{R}$. As in Sec. I.2 for given τ we denote by $\Lambda_\tau(t)$ the identical embedding of $\Lambda(t)$ in the affine space $\Lambda(\tau)^{\pitchfork}$. Fixing an "origin" in $\Lambda(\tau)^{\pitchfork}$ we make $\Lambda_\tau(t)$ a vector function with values in the space of self-adjoint linear mappings from $\Lambda(\tau)^*$ to $\Lambda(\tau)$. Such a vector function has the pole at $t = \tau$. Assume that the order of the pole at $t = \tau$ is equal to $l(\tau)$.

Definition 1. The curve $\Lambda : I \mapsto L(W)$ of the constant (finite) weight is called flat at the point $\tau \in I$, if the function $t \mapsto \Lambda_\tau(t)$ satisfies

$$\Lambda_\tau(t) = \Lambda^0(\tau) + \sum_{i=-l(\tau)}^{-1} Q_i(\tau)(t-\tau)^i. \tag{1.1}$$

2000 *Mathematics Subject Classification.* 53A55, 37J99, 70G45, 93B29.

Key words and phrases. Lagrange Grassmannian, symplectic invariants, flat curves, conjugate points, comparison theorems.

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Definition 2. The curve $\Lambda : I \mapsto L(W)$ of constant (finite) weight is called flat if it is flat at any point of I.

First, consider the regular curves (recall that the curve $\Lambda(\cdot)$ is said to be regular if the velocity $\dot\Lambda(t)$ is a nondegenerate quadratic form for any t, see Sec. I.3 for details). In this case the function $t \mapsto \Lambda_\tau(t)$ has a simple pole at $t = \tau$ for any τ. Then, by definition, the regular curve $\Lambda(\cdot)$ is flat at τ iff

$$\Lambda_\tau(t) = \Lambda^0(\tau) + Q_{-1}(\tau)\frac{1}{t-\tau}. \tag{1.2}$$

The coordinate version of (1.2) is

$$(S_t - S_\tau)^{-1} = A_0(\tau) + \frac{\dot S_\tau^{-1}}{t-\tau}, \tag{1.3}$$

where $\Lambda(t) = \{(x, S_t x) : x \in \mathbb{R}^m\}$ (see relation (3.2) of Part I). Suppose that the positive index of $\dot\Lambda(t)$ is equal to p (this index is constant, since the rank is constant). Denote by $I^{\pm}_{p,s}$ the following matrix

$$I^{\pm}_{p,s} = \begin{pmatrix} I_p & 0 \\ 0 & -I_s \end{pmatrix}, \tag{1.4}$$

where I_n is the $n \times n$-identity matrix. One can choose the coordinates such that $S_\tau = 0$, $A_0(\tau) = 0$, and $\dot S_\tau = I^{\pm}_{p,m-p}$. Substituting this in (1.3), we obtain the following lemma.

Lemma 1.1. *If the regular curve $\Lambda : I \in \mathbb{R} \mapsto L(W)$ with the positive index p of $\dot\Lambda(t)$ is flat at some $\tau \in I$, then the curve $\Lambda(t)$ has the coordinate representation S_t, $\Lambda(t) = \{(x, S_t x) : x \in \mathbb{R}^m\}$ such that*

$$S_t = I^{\pm}_{p,m-p}(t-\tau). \tag{1.5}$$

This lemma implies the following proposition.

Proposition 1. *If the regular curve $\Lambda : I \in \mathbb{R} \mapsto L(W)$ with the positive index p of $\dot\Lambda(t)$ is flat at some $\tau \in I$, then it is flat. Moreover, the curve $\Lambda(t)$ has the coordinate representation S_t, $\Lambda(t) = \{(x, S_t x) : x \in \mathbb{R}^m\}$ such that*

$$S_t = I^{\pm}_{p,m-p}t. \tag{1.6}$$

i.e., the curve $\bar{\Lambda}(t)$ is flat at any $\bar{\tau} \in \mathbb{R}$. In particular, $\bar{\Lambda}(t)$ is flat at 0. So, by Lemma 1.1, the curve $\Lambda(t)$ has the coordinate representation S_t, satisfying (1.6). □

Remark 1. Recall that the transition functions from one chart to another in the Lagrange Grassmannian $L(W)$ are "matrix Möbius transformations" of the type $S \mapsto (C + DS)(A + BS)^{-1}$ with

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}(2m). \tag{1.7}$$

This together with (1.6) implies that the regular curve $\Lambda : I \in \mathbb{R} \mapsto L(W)$ with the positive index p of $\dot{\Lambda}(t)$ is flat iff any its coordinate representation S_t, $\Lambda(t) = \{(x, S_t x) : x \in \mathbb{R}^m\}$ has the following form

$$S_t = (C + DI^{\pm}_{p,m-p}t)(A + BI^{\pm}_{p,m-p}t)^{-1}, \tag{1.8}$$

where $m \times m$ matrices A,B, C, and D satisfy (1.7).

Evidently, the direct sum of flat curves is a flat curve. Namely, suppose that $W = W_1 \oplus W_2$, where W_i, $i = 1, 2$, are symplectic spaces such that symplectic forms on each W_i are restrictions of the symplectic form on W. If curves $\Lambda_i : I \mapsto L(W_i)$ are flat, then the curve

$$\Lambda : I \mapsto L(W), \quad \Lambda(t) = \Lambda_1(t) \oplus \Lambda_2(t),$$

is flat. So, one can construct flat curves of arbitrary rank $r < m$ by taking direct sum of rank 1 flat curves.

Now we will give models for rank 1 flat curves. Let $\mathcal{V}_m(t)$ be the vector in $\mathbb{R}^m$ with ith component $(\mathcal{V}_m(t))_i = t^{i-1}$. Let $F_m(t)$ be $m \times m$ symmetric matrix such that

$$F_m(t) = \int_0^t \mathcal{V}_m(\xi)^T \mathcal{V}_m(\xi)\, d\xi = \left\{ \frac{t^{i+j-1}}{i+j-1} \right\}_{i,j=1}^m. \tag{1.9}$$

Proposition 2. *If the curve $\Lambda : I \mapsto L(W)$ in some coordinates has the form*

$$\Lambda(t) = \{(x, \pm F_m(t)x) : x \in \mathbb{R}^m\}, \tag{1.10}$$

then $\Lambda(\cdot)$ is flat.

Proof. Evidently, it is sufficient to prove the Proposition in the case, when the sign $+$ appears in (1.10). By construction, $\Lambda(\cdot)$ is a rank 1 curve. So, its weight is not less than m^2 (see, for example, Lemma I.6.1). On the other hand, by (1.9) the function $(t, \tau) \mapsto \det(F_m(t) - F_m(\tau)$ is a polynomial of order not greater than m^2. It implies that the weight of $\Lambda(\cdot)$ at any point is equal to m^2 and by definition of the weight the function $X(t, \tau) =$

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$$\det(F_m(t) - F_m(\tau)) = C(t-\tau)^{m^2}. \tag{1.11}$$

Moreover, it is clear that all entries of the adjacency matrix of the matrix $F_m(t) - F_m(\tau)$ are polynomials of degree less than m^2. This together with (1.11) implies that for any τ the matrix function $t \mapsto (F_m(t) - F_m(\tau))^{-1}$ coincides with the principal part of its Laurent expansion at τ. But this is exactly the coordinate version of (1.1), which implies that $\Lambda(\cdot)$ is flat. □

Definition 3. The curve $\Lambda : I \mapsto L(W)$ is said to be strongly flat if it can be represented as a direct sum of curves of the form (1.10).

Later on we will prove that the rank 1 flat curves of the form (1.10) are the only flat curves of rank 1, up to a symplectic transformation. Therefore, the strong flat curves are the curves which can be represented as direct sums of r rank 1 flat curves.

Remark 2. From (1.6) it follows that if a regular curve is flat then it is strongly flat.

It is not clear yet, whether in general any flat curve is strong flat (this is a part of our general conjecture formulated below).

Now we will prove three general propositions related to the flat curves.

Proposition 3. *Let* $\Lambda : I \mapsto L(W)$ *be a flat curve of weight* k. *Then the function* $g(t, \tau)$ *is identically equal to zero for any* $t, \tau \in I$, *or, equivalently, the remarkable identity*

$$\det\left([\Lambda(t_0), \Lambda(t_1), \Lambda(t_2), \Lambda(t_3)]\right) = [t_0, t_1, t_2, t_3]^k \tag{1.12}$$

holds for any four parameters $t_0, t_1, t_2, t_3 \in I$.

Proof. From Lemma I.4.2, it follows (see relations (4.19) and (4.23) of Part I) that

$$\operatorname{tr}\left(\frac{\partial}{\partial t}\Lambda_\tau(t) \circ \dot{\Lambda}(\tau)\right) = -\frac{k}{(t-\tau)^2} - g(t, \tau). \tag{1.13}$$

On the other hand, by assumption, relation (1.1) holds for all $\tau \in I$. Therefore,

$$\operatorname{tr}\left(\frac{\partial}{\partial t}\Lambda_\tau(t)\circ\dot\Lambda(\tau)\right) = \\ = \sum_{i=-l-1}^{-2}(i+1)\operatorname{tr}\left(Q_{i+1}(\tau)\circ\dot\Lambda(\tau)\right)(t-\tau)^i. \tag{1.14}$$

This and (1.13) imply that

$$g(t,\tau) = 0 \ \ \forall t,\tau \in I. \tag{1.15}$$

The proof of the proposition is completed. □

Proposition 4. *Let $\Lambda : I \mapsto L(W)$ be a flat curve. Then the derivative curve $\Lambda^0(\cdot)$ of the curve $\Lambda(\cdot)$ is constant, i.e.,*

$$\dot\Lambda^0(\tau) = 0 \ \ \forall \tau \in I. \tag{1.16}$$

Proof. Take $\bar\tau \in I$. Let S_t, $\Lambda(t) = \{(x, S_t\, x) : x \in \mathbb{R}^m\}$, be a coordinate representation of the germ of $\Lambda(t)$ at $t = \bar\tau$ such that $\Lambda^0(\bar\tau) = 0 \oplus \mathbb{R}^n$. By the assumption,

$$(S_t - S_\tau)^{-1} = \sum_{i=-l}^{0} A_i(\tau)(t-\tau)^i, \tag{1.17}$$

and by construction,

$$A_0(\bar\tau) = 0. \tag{1.18}$$

Differentiating both sides of (1.17) w.r.t. τ at $\tau = \bar\tau$, comparing free terms of the corresponding expansions and using (1.18), we obtain $\dot A_0(\bar\tau) = 0$ (see also recursive formula (2.4) from Part I in the case $i = 0$). But this means that $\dot\Lambda^0(\bar\tau) = 0$. □

From Propositions 2, 3, and 4 it immediately follows that

Proposition 5. *For strong flat curves $\Lambda : I \mapsto L(W)$, identities* (1.12) *and* (1.16) *are valid.*

Now we will prove that for the regular and rank 1 curves all properties of flat curves mentioned above are equivalent.

Theorem 1. *Let curve $\Lambda : I \mapsto L(W)$ be a regular curve or rank* 1 *curve of constant* (*finite*) *weight. Then the following five statements are equivalent*:

(1) *the curve $\Lambda(\cdot)$ is flat at some $\tau \in I$*;
(2) *the curve $\Lambda(\cdot)$ is flat*;

(3) *the function* $g(t,\tau)$ *is equal identically to zero for any* $t,\tau\in I$, *or, equivalently, the remarkable identity*

$$\det\Big([\Lambda(t_0),\Lambda(t_1),\Lambda(t_2),\Lambda(t_3)]\Big)=[t_0,t_1,t_2,t_3]^k \qquad (1.19)$$

holds for any four parameters t_0, t_1, t_2, $t_3\in I$, *where* k *is the weight of the curve* ($k=\frac{1}{2}\dim W$ *in the regular case and* $k=\Big(\frac{1}{2}\dim W\Big)^2$ *in the rank* 1 *case*);

(4) *the derivative curve* $\Lambda^0(\cdot)$ *of the curve* $\Lambda(\cdot)$ *is constant, i.e.,*

$$\dot{\Lambda}^0(t)=0\ \ \forall t\in I; \qquad (1.20)$$

(5) *the curve* $\Lambda(\cdot)$ *is strongly flat.*

Before starting the proof of the Theorem 1 we give several its direct consequences. As in [1], let

$$\beta_{0,i}(\tau)=\frac{1}{i!}\,\frac{\partial^i g}{\partial\tau^i}(t,\tau)\Big|_{t=\tau}. \qquad (1.21)$$

Recall that Theorem I.2 states that the coefficients $\beta_{0,2k}(\tau)$, $0\le k\le m-1$, constitute a complete system of symplectic invariants of the curve $\Lambda(\tau)$ of rank 1 and constant weight, i.e., determine $\Lambda(\tau)$ uniquely, up to a symplectic transformation.

Corollary 1. *The rank* 1 *curve* $\Lambda: I\mapsto L(W)$ *of constant weight is flat iff all functions* $\beta_{0,2k}(\tau)$ *with* $0\le k\le m-1$ *are identically equal to zero, i.e., rank* 1 *flat curves are the curves with vanishing complete system of symplectic invariants.*

Proof. Indeed, if the curve $\Lambda(\cdot)$ is flat then by Theorem 1, $g(t,\tau)\equiv 0$, which trivially implies that $\beta_{0,2k}(\tau)\equiv 0$, $0\le k\le m-1$. On the other hand, by Proposition 5 and Theorem I.2 the strong flat curves are the only curves with $\beta_{0,2k}(\tau)\equiv 0$, $0\le k\le m-1$, which proves sufficiency. □

Remark 3. By Corollary 1, the invariants $\beta_{0,i}(\tau)$ play the role of curvatures: the curve is flat iff all these invariants vanish.

Corollary 2. *There is an embedding of the real projective line* $\mathbb{RP}^1$ *into* $L(W)$ *as a closed rank* 1 *flat curve endowed with the canonical projective structure*; *the Maslov index of this curve equals* m. *All other rank* 1 *flat curves are images under symplectic transformations of* $L(W)$ *of the segments of this unique one rank* 1 *flat curve.*

Proof. By Theorem 1 any rank 1 flat curve $\Lambda : I \mapsto L(W)$ is strongly flat, i.e., in some coordinates it satisfies (1.10). Equation (1.10) defines the flat extension $\bar\Lambda(\cdot)$ of $\Lambda(\cdot)$ to the whole $\mathbb{R}$. By Proposition 5, the derivative curve of $\bar\Lambda(\cdot)$ is constant. Suppose that $\bar\Lambda^0(t) \equiv \Lambda^0$. From the proof of Proposition 2 it immediately follows that $\Lambda^0 = 0 \oplus \mathbb{R}^m$. This and the fact that $\lim_{t\to\infty} F_m(t) = \infty$ imply that

$$\lim_{t\to\infty} \Lambda(t) = \Lambda^0, \tag{1.22}$$

i.e., the curve $\bar\Lambda$ can be extended continuously to (closed) curve defined on $\mathbb{RP}^1$. We will denote this extension also by $\bar\Lambda(\cdot)$. Note that from definition it easily follows that if $\Lambda : I \mapsto L(W)$ is a flat curve, then any its reparametrization by a Möbius transformation is also flat, i.e., for any Möbius transformation $\varphi(\tau)$ the curve $\tau \mapsto \Lambda(\varphi(\tau))$ is flat on $\varphi^{-1}(I)$. In particular, the curve $t \mapsto \bar\Lambda\left(\dfrac{1}{t}\right)$ is flat and, therefore, by Theorem 1, is also strongly flat on $\mathbb{R}\backslash 0$. This implies that $\bar\Lambda$ defines a smooth embedding of $\mathbb{RP}^1$ into $L(W)$. Obviously, Maslov index of this embedding equals m. Also, we note also that the original parameter on $\bar\Lambda$ is projective w.r.t. the canonical projective structure on $\bar\Lambda(\cdot)$, since by Corollary 1 the Ricci curvature of rank a 1 flat curve is equal to zero. This completes the proof. □

Since any strong flat curve is a direct sum of rank 1 flat curves, we have the following corollary.

Corollary 3. *There is an embedding of the real projective line $\mathbb{RP}^1$ into $L(W)$ as a strongly flat closed curve endowed with the canonical projective structure; the Maslov index of this curve equals m. All other strongly flat curves are images under symplectic transformations of $L(W)$ of the segments of this unique strongly flat curve.*

Remark 4. Obviously, item 2 of Theorem 1 implies item 1. Proposition 3 means that $2 \Rightarrow 3$. Proposition 4 means that $2 \Rightarrow 4$. Proposition 5 means that $5 \Rightarrow 2$. Theorem 1 will be proved, if one proves, for example, that $1 \Rightarrow 5$, $4 \Rightarrow 2$ (or 5), and $3 \Rightarrow 2$.

Proof of Theorem 1.

1. The case of regular curves. Recall that the curvature operator $R(t) : \Lambda(t) \to \Lambda(t)$ of the curve $\Lambda(\cdot)$ is defined as follows:

$$R(t) = -\dot\Lambda^0(t) \circ \dot\Lambda(t). \tag{1.23}$$

By (1.23) and Proposition 4, for any flat curve $\Lambda(t)$ the curvature operator is identically equal to zero. For the regular curves, the opposite statement is true.

Lemma 1.2. *If the curvature operator of a regular curve is identically equal to zero, then the curve is flat.*

Proof. According to Sec. 3 of [1] (see also [2]), for any coordinate representation S_t of the curve $\Lambda(t)$, the matrix

$$\mathbb{S}(S_t) = \frac{d}{dt}\Big((2\dot{S}_t)^{-1}\ddot{S}_t\Big) - \Big((2\dot{S}_t)^{-1}\ddot{S}_t\Big)^2 \tag{1.24}$$

represents the curvature operator $R(t)$ in some basis. Therefore, $R(t) \equiv 0$ iff $\mathbb{S}(S_t) \equiv 0$. Note that $\mathbb{S}(S_t)$ is a matrix analog of the Schwarz derivative and it has a similar property: $\mathbb{S}(S_t) \equiv 0$ iff $S_t = (C + Dt)(A + Bt)^{-1}$, where $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{GL}(2m)$ (see, e.g., [7], p. 205). One can suppose in the beginning that $S_\tau = 0$ for some parameter τ. Then $C = -D\tau$ and

$$(S_t - S_\tau)^{-1} = \frac{1}{t-\tau}(A + Bt)D^{-1},$$

which implies the flatness of $\Lambda(\cdot)$ at τ and, therefore, by Proposition 1, flatness of $\Lambda(\cdot)$. □

Using the previous lemma one can easily prove that $4 \Rightarrow 2$. Indeed, if $\dot{\Lambda}(t)^0 \equiv 0$, then by (1.23) $R(t) \equiv 0$ and, therefore, by Lemma 1.2 the curve $\Lambda(\cdot)$ is flat. Also, Remark 2 means that $1 \Rightarrow 5$. By Remark 4, in order to prove Theorem 1 in the regular case it remains to show that $3 \Rightarrow 2$. It turns out that the following even more strong statement holds.

Lemma 1.3. *If $\Lambda : I \mapsto L(W)$ is a regular curve such that its Ricci curvature and fundamental form are identically equal to zero, then the curve $\Lambda(\cdot)$ is flat.*

Proof. If the Ricci curvature of a regular curve $\Lambda(t)$ is identically equal to zero, then by (4.27) from Part I the curvature operator $R(t)$ of $\Lambda(t)$ satisfies

$$\operatorname{tr} R(t) \equiv 0. \tag{1.25}$$

If also the fundamental form of the regular curve $\Lambda(t)$ is identically equal to zero, then substituting (1.25) into (5.12) from Part I, we obtain

$$\operatorname{tr}\Big(R(t)^2\Big) \equiv 0. \tag{1.26}$$

Note that by definition, $R(t) : \Lambda(t) \mapsto \Lambda(t)$ is a linear operator, symmetric w.r.t. (pseudo)-Euclidean structure, defined by the quadratic form $\dot{\Lambda}(t)$. Therefore, Eq. (1.26) implies that

$$R(t) \equiv 0.$$

Hence, by Lemma 1.2, the curve $\Lambda(\cdot)$ is flat. □

The proof of Theorem 1 in the case of regular curves is completed.

2. The case of rank 1 curves. Without loss of generality, it can be assumed that the curve $\Lambda(\cdot)$ is nondecreasing.

2.1. The proof of implications $3 \Rightarrow 2$ **and** $1 \Rightarrow 5$**.** As in Sec. I.6, we denote by $w(t,\tau)$ the vector in $\Lambda(\tau)$ such that for any $v \in \Lambda(\tau)^*$

$$\left\langle v, \frac{\partial}{\partial t}\Lambda_\tau(t)v \right\rangle = -\langle v, w(t,\tau)\rangle^2. \tag{1.27}$$

Let $w_i(t,\tau)$ be the i-th component of $w(t,\tau)$ w.r.t. the canonical basis $e_1(\tau), \dots e_m(\tau)$ of the subspace $\Lambda(\tau)$ (see formulas (6.2)–(6.5) from Part I for the definition of the canonical basis; see also Proposition I.4 which corresponds to the case of the curve of constant rank). From Proposition I.4 and Corollary I.2 it follows that the functions $w_i(t,\tau)$ have the form

$$w_i(t,\tau) = \frac{1}{(t-\tau)^{m-i+1}} + \varphi_i(t,\tau), \tag{1.28}$$

where $\varphi_i(t,\tau)$ are smooth functions. We prove the following lemma.

Lemma 1.4. *If $\Lambda : I \mapsto L(W)$ is a rank 1 curve, which is flat at $\tau \in I$, then the functions $t \mapsto w_i(t,\tau)$ satisfy*

$$w_i(t,\tau) = \frac{1}{(t-\tau)^{m-i+1}}. \tag{1.29}$$

Proof. Using the same arguments as in the proof of Proposition 3 (formulas (1.13) and (1.14)), we obtain that if $\tau \in I$ is a parameter such that (1.1) holds, then

$$g(t,\tau) = 0 \ \forall t \in I. \tag{1.30}$$

Recall that the functions $g(t,\tau)$ and $w_m(t,\tau)$ are related by the following identity (see Lemma I.7.1):

$$w_m^2(t,\tau) = \frac{1}{(t-\tau)^2} + \frac{1}{m^2}g(t,\tau). \tag{1.31}$$

Hence, by (1.30),

$$w_m(t,\tau) = \frac{1}{t-\tau}. \tag{1.32}$$

Let S_t, $\Lambda(t) = \{(x, S_t\, x) : x \in \mathbb{R}^m\}$, be a coordinate representation of a germ of $\Lambda(t)$ at $t = \tau$ such that the canonical basis $e_1(\tau), \dots, e_m(\tau)$ is a standard basis of $\mathbb{R}^m$. Since $\Lambda(t)$ is flat, we have the following coordinate version of expansion (1.1):

$$(S_t - S_\tau)^{-1} = \sum_{i=-l}^{0} A_i(\tau)(t-\tau)^i. \tag{1.33}$$

Therefore,

$$\frac{\partial}{\partial t}(S_t - S_\tau)^{-1} = \sum_{i=-l-1}^{-2} (i+1)\, A_{i+1}(\tau)(t-\tau)^i. \tag{1.34}$$

On the other hand,

$$\left(\frac{\partial}{\partial t}(S_t - S_\tau)^{-1}\right)_{i,j} = -w_i(t,\tau)w_j(t,\tau). \tag{1.35}$$

Therefore,

$$\begin{aligned}\left(\frac{\partial}{\partial t}(S_t - S_\tau)^{-1}\right)_{m,i} &= -\frac{1}{t-\tau}\left(\frac{1}{(t-\tau)^{m-i+1}} + \varphi_i(t,\tau)\right) = \\ &= -\frac{1}{(t-\tau)^{m-i+2}} - \frac{\varphi_i(t,\tau)}{t-\tau}.\end{aligned} \tag{1.36}$$

Comparing (1.34) and (1.36), one can easily conclude that $\varphi_i(t,\tau) = 0$ for any $t \in I$, which implies (1.29). The proof of the lemma is completed. □

From the proof of the previous lemma it follows that if $g(t,\tau) \equiv 0$, then relations (1.29) are valid for any $t, \tau \in I$, which, in turn, implies that (1.1) holds for any $\tau \in I$ (see, e.g., (1.27) and (1.35)). By this, we actually have proved that *item* 3 *of Theorem* 1 *implies item* 2.

To show that $1 \Rightarrow 5$, we prove first the following lemma.

Lemma 1.5. *If $\Lambda : I \mapsto L(W)$ is nondecreasing rank 1 curve, which is flat at $\tau \in I$, then in some coordinates the curve $\Lambda(\cdot)$ has the form*

$$\Lambda(t) = \{(x, F_m(t-\tau)x)\, x \in \mathbb{R}^m\}, \tag{1.37}$$

where $F_m(t)$ is defined by (1.9).

Proof. We take again some coordinate representation S_t, $\Lambda(t) = \{(x, S_t\, x) : x \in \mathbb{R}^m\}$, of the germ of $\Lambda(t)$ at $t = \tau$ such that the canonical basis $e_1(\tau), \dots, e_m(\tau)$ is a standard basis of $\mathbb{R}^m$. Then, obviously, $S_\tau = 0$. By the previous lemma,

$$\left((S_t)^{-1}\right)_{i,j} = \frac{1}{(2m-i-j+1)(t-\tau)^{2m-i-j+1}}. \tag{1.38}$$

Then, referring to the formulas (6.16) and (6.17) from Part I (with $k_i = i-1$ which corresponds to the case of constant weight), we have

$$(S_t)_{i,j} = \int_\tau^t v_i(\xi) v_j(\xi)\, d\xi, \tag{1.39}$$

where

$$v_i(t) = c_i (t-\tau)^{m-i}, \quad c_i \neq 0. \tag{1.40}$$

It is easy to see that if $\bar{S}_t$, $\Lambda(t) = \{(x, \bar{S}_t\, x) : x \in \mathbb{R}^m\}$, is a coordinate representation of the germ of $\Lambda(t)$ at $t = \tau$ such that the basis $\left(\frac{e_m(\tau)}{c_m}, \frac{e_{m-1}(\tau)}{c_{m-1}}, \ldots, \frac{e_1(\tau)}{c_1}\right)$ is a standard basis of $\mathbb{R}^m$, then $\bar{S}_t = F_m(t-\tau)$. This completes the proof. □

Therefore, if $\Lambda : I \mapsto L(W)$ is a nondecreasing rank 1 curve, which is flat at $\tau \in I$, then formula (1.37) defines the extension $\bar{\Lambda}(\cdot)$ of $\Lambda(\cdot)$ to the whole $\mathbb{R}$. Hence from Proposition 2 it easily follows that for the curve $\bar{\Lambda}(\cdot)$, relation (1.1) holds for all real τ. Therefore, one can use the previous Lemma for all τ, in particular, for $\tau = 0$. Therefore, there exists a coordinate representation of $\Lambda(\cdot)$ satisfying (1.10), i.e., the curve $\Lambda(\cdot)$ is strongly flat. We have proved that *item 1 of Theorem 1 implies item 5.*

2.2. The proof of the implication $4 \Rightarrow 5$**.** According to Remark 4, in order to complete the proof of Theorem 1 for rank 1 curves it is sufficient to prove that $4 \Rightarrow 5$.

Let $\beta_{0,k}(t)$ be as in (1.21). Note that in order to prove that the curve is strongly flat it is sufficient to show that

$$\beta_{0,2i}(t) \equiv 0, \quad 0 \le i \le m-1. \tag{1.41}$$

Indeed, by Proposition 5 and Theorem I.2 the strongly flat curves are the only curves, satisfying (1.41). Therefore, the implication $4 \Rightarrow 5$ follows from the following proposition.

Proposition 6. *If a rank 1 curve* $\Lambda : I \mapsto L(W)$ *of constant weight satisfies*

$$\dot{\Lambda}^0(t) \equiv 0, \tag{1.42}$$

then relations (1.41) *are valid.*

The rest of the section is devoted to the proof of Proposition 6.

Proof of Proposition 6. Let, as before, $(e_1(\tau), \dots, e_m(\tau))$ be a canonical basis of $\Lambda(t)$ and $(f_1(t), \dots, f_m(\tau))$ be a basis of $\Lambda^0(t)$ dual to $(e(\tau), \dots, e_m(\tau))$, i.e., $\sigma(f_i(\tau), e_i(\tau)) = \delta_{i,j}$ (in Sec. I.7 the basis $(e_1(\tau), \dots, e_m(\tau), f_1(\tau), \dots, f_m(\tau))$ of W was called the canonical moving frame associated with the curve $\Lambda(\cdot)$). Let S_t^0 be the coordinate representation of the derivative curve $\Lambda^0(t)$, $\Lambda^0(t) = \{(x, S_t^0 x) : x \in \mathbb{R}^m\}$ such that $(f_1(\tau), \dots, f_m(\tau))$ is a standard basis of $\mathbb{R}^m \oplus 0$ and $(e_1(\tau), \dots, e_m(\tau))$ is a standard basis of $0 \oplus \mathbb{R}^m$.

It is convenient to introduce the following notation: for given tuple $\{\psi_i(t)\}_{i=1}^N$ of smooth functions on I we will denote by $\mathrm{Pol}\big(\{\psi_i(\tau)\}_{i=1}^N\big)$ any function on I, which can be expressed as a polynomial without free term w.r.t. the functions $\psi_i(\tau)$, $1 \le i \le N$, and their derivatives. We claim that in order to prove (1.41), it is sufficient to prove the following two lemmas.

Lemma 1.6. *The (i,j)th entry $\left(\dot S_\tau^0\right)_{i,j}$ of the matrix $\dot S_\tau^0$ with even $i+j$ can be represented in the form*

$$\left(\dot S_\tau^0\right)_{i,j} = c_{i,j}\beta_{0,2(m-\frac{i+j}{2})}(\tau) + \mathrm{Pol}\left(\{\beta_{0,2s}(\tau)\}_{s=0}^{m-\frac{i+j}{2}-1}\right), \quad (1.43)$$

where $c_{i,j}$ is some constant.

Lemma 1.7. *The constant $c_{i,j}$ from* (1.43) *can be chosen such that*

$$c_{m,j} \neq 0, \quad 2 \le j \le m, \quad m+j \ \textit{ is even}, \quad (1.44)$$

$$c_{i,2} \neq 0, \quad 2 \le i \le m-1 \ \textit{ is even}, \quad (1.45)$$

$$c_{1,1} \neq 0. \quad (1.46)$$

Indeed, assume that (1.42) holds. Recall that the velocity $\dot\Lambda^0(\tau)$ can be considered as a self-adjoint linear mapping from $\Lambda^0(\tau)$ to $\Lambda^0(\tau)^*$. Fixing a basis $\big(f_1(\tau), \dots, f_m(\tau)\big)$ in $\Lambda^0(\tau)$, we identify this mapping with a symmetric matrix. Then under this identification

$$\dot\Lambda^0(\tau) = -\dot S_\tau^0. \quad (1.47)$$

If Lemma 1.6 holds, then from (1.42), (1.43), and (1.47) it follows that

$$c_{i,j}\beta_{0,2(m-\frac{i+j}{2})}(\tau) + \mathrm{Pol}\left(\{\beta_{0,2s}(\tau)\}_{s=0}^{m-\frac{i+j}{2}-1}\right) = 0. \quad (1.48)$$

If Lemma 1.7 also holds, then applying (1.48) first for $i = j = m$ and using (1.44), we obtain $\beta_{0,0}(\tau) \equiv 0$. It is obvious that if $\beta_{0,2s}(\tau) \equiv 0$ for all $0 \le s \le m - \dfrac{i+j}{2} - 1$, then

$$\mathrm{Pol}\left(\{\beta_{0,2s}(\tau)\}_{s=0}^{m-\frac{i+j}{2}-1}\right) \equiv 0.$$

Using this fact and one of relations (1.44)–(1.46), we obtain from (1.48) by induction that $\beta_{0,2s}(\tau) \equiv 0$ for all $0 \le s \le m-1$. Therefore, Lemmas 1.6 and 1.7 imply Proposition 6.

Remark 5. Actually, we believe that for all pairs of indices (i,j) with even $i+j$ one can take constants $c_{i,j} \neq 0$ in (1.43) but for our purposes it was sufficient to verify that for any $0 \le k \le m-1$ among all pairs of indices $\{(i,j) : 2m-i-j = 2k\}$ there exists at least one (i_k, j_k) such that $c_{i_k,j_k} \neq 0$. In Lemma 1.7 we verify the last assertion by choosing pairs (i_k, j_k) for which the calculation of c_{i_k,j_k} is relatively easy.

Now we prove Lemmas 1.6 and 1.7.

Proof of Lemma 1.6. Let S_t be the coordinate representation of the curve $\Lambda(t)$, $S_t = \{(x, S_t x) : x \in \mathbb{R}^m\}$ such that $(e_1(\tau), \dots, e_m(\tau))$ is a standard basis of $\mathbb{R}^m \oplus 0$ and $(f_1(\tau), \dots, f_m(\tau))$ is a standard basis of $0 \oplus \mathbb{R}^m$. Suppose, as in Sec. I.2 that the function $t \mapsto (S_t - S_\tau)^{-1}$ has the Laurent expansion

$$(S_t - S_\tau)^{-1} \approx \sum_{i=-l}^{\infty} A_i(\tau)(t-\tau)^i \tag{1.49}$$

at $t = \tau$. Applying recursive formula (2.4) from Part I for $i = 0$, one can express $\dot{S}^0_\tau$ as follows:

$$\begin{aligned} \dot{S}^0_\tau = \frac{d}{d\tau} A_0(\tau) = A_1(\tau) + \\ + \sum_{n=-l}^{-1} \Big(A_n(\tau)\dot{S}_\tau A_{-n}(\tau) + A_{-n}(\tau)\dot{S}_\tau A_n(\tau) \Big). \end{aligned} \tag{1.50}$$

On the other hand,

$$\Big((S_t - S_\tau)^{-1} \Big)_{i,j} = -\int^t w_i(\xi,\tau) w_j(\xi,\tau) d\,\xi. \tag{1.51}$$

Therefore, by (1.28), the order l of the pole in (1.49) satisfies $l = 2m-1$. Recall that, according to (7.3) from Part I,

$$(\dot{S}_\tau))_{i,j} = \begin{cases} 0 & (i,j) \neq (m,m), \\ m^2 & (i,j) = (m,m). \end{cases} \tag{1.52}$$

Using formulas (1.50)–(1.52), we will compute $\dot{S}^0_\tau$. First, by (1.52), we obtain

$$(A_n(\tau)\dot{S}_\tau A_{-n}(\tau))_{i,j} = m^2 (A_n)_{i,m} (A_{-n})_{m,j}. \tag{1.53}$$

Substituting (1.53) into (1.50), we obtain

$$\left(\dot S^0_\tau\right)_{i,j} = (A_1)_{i,j} + \\ + m^2 \sum_{n=1-2m}^{-1} \Big((A_n)_{i,m}(A_{-n})_{m,j} + (A_{-n})_{i,m}(A_n)_{m,j}\Big). \qquad (1.54)$$

Let $\varphi_i(t,\tau)$ be as in (1.28). Denote

$$\varphi_{i,j}(\tau) = \frac{1}{j!}\frac{\partial^j}{\partial t^j}\varphi_i(t,\tau)\Big|_{t=\tau} \qquad (1.55)$$

From (1.51) and (1.28) it is not difficult to obtain the following three relations:

$$(A_{-n})_{m,j} = \begin{cases} 0, & j > m-n+1, \\ \frac{1}{n}, & j = m-n+1, \\ \frac{1}{n}, \varphi_{m-n-j} & 1 \le j < m-n+1, \end{cases} \qquad (1.56)$$

where $1 \le n \le 2m-1$,

$$(A_n)_{i,m} = -\frac{1}{n}\left(\varphi_{m,n+m-i} + \varphi_{i,n}\right) - \frac{1}{n}\sum_{k=0}^{n-1}\varphi_{i,k}\varphi_{m,n-1-k}, \qquad (1.57)$$

where $n \ge 1$, and

$$(A_1)_{i,j} = -\varphi_{j,m-i+1} - \varphi_{i,m-j+1}. \qquad (1.58)$$

Also, we note that from (1.51) it follows that the coefficient of $\frac{1}{t-\tau}$ in the expansion of $t \mapsto w_i(t,\tau)w_j(t,\tau)$, $1 \le i,j \le m$, in the formal Laurent series at $t=\tau$ is equal to zero, otherwise the expansion of $t \mapsto (S_t - S_\tau)^{-1}$ contains a logarithmic term. Hence we have the relation

$$\varphi_{i,m-j} + \varphi_{j,m-i} = 0 \qquad (1.59)$$

which will be used in the sequel.

Substituting (1.56)–(1.58) in (1.54), we have

$$(\dot S^0_\tau)_{i,j} = \Upsilon_{i,j}(\tau) + \Theta_{i,j}(\tau), \qquad (1.60)$$

where

$$\Upsilon_{i,j} = -\left(1 + \frac{m^2}{(m-i+1)^2}\right)\varphi_{j,m-i+1} - \left(1 + \frac{m^2}{(m-j+1)^2}\right)\varphi_{i,m-j+1} - \\ - m^2\left(\frac{1}{(m-i+1)^2} + \frac{1}{(m-j+1)^2}\right)\varphi_{m,2m-i-j+1} \qquad (1.61)$$

and

$$\Theta_{i,j}(\tau) = -m^2\Bigg(\frac{1}{(m-j+1)^2}\sum_{k=0}^{m-j}\varphi_{i,k}\varphi_{m,m-j-k} + \\ + \frac{1}{(m-i+1)^2}\sum_{k=0}^{m-i}\varphi_{i,k}\varphi_{m,m-i-k} + \\ + \sum_{n=1}^{m-j}\frac{1}{n^2}\Bigg(\varphi_{m,n+m-1} + \varphi_{i,n} + \\ + \sum_{k=0}^{n-1}\varphi_{i,k}\varphi_{m,n-1-k}\Bigg)\varphi_{m,m-n-j} + \\ + \sum_{n=1}^{m-j}\frac{1}{n^2}\Bigg(\varphi_{m,n+m-1} + \varphi_{j,n} + \\ + \sum_{k=0}^{n-1}\varphi_{j,k}\varphi_{m,n-1-k}\Bigg)\varphi_{m,m-n-i}\Bigg). \tag{1.62}$$

Similarly to the proof of Theorem I.2 in Sec. I.7, we denote

$$u_i(t,\tau) = (t-\tau)^{m-i+1}w_i(t,\tau). \tag{1.63}$$

Let

$$u_{i,k}(\tau) = \frac{1}{k!}\frac{\partial^k}{\partial t^k}u_i(t,\tau)\Big|_{t=\tau} \tag{1.64}$$

Note that by (1.28),

$$u_{i,0}(\tau) \equiv 1, \quad 1 \le i \le m, \tag{1.65}$$
$$u_{i,k}(\tau) \equiv 0, \quad 1 \le k \le m-i, \quad 1 \le i \le m. \tag{1.66}$$

We claim that in order to prove Lemma 1.6 it is sufficient to show that

$$u_{i,k}(\tau) = \begin{cases} \mathrm{Pol}\left(\left\{u_{m,2s}(\tau)\right\}_{s=1}^{\min\{[\frac{k}{2}],m\}}\right), & k \notin \{2s : 1 \le s \le m\},\ k > 0, \\ \mu_{i,k}u_{m,k}(\tau) + \\ \quad + \mathrm{Pol}\left(\left\{u_{m,2s}(\tau)\right\}_{s=1}^{\frac{k}{2}-1}\right), & k \in \{2s : 1 \le s \le m\}, \end{cases} \tag{1.67}$$

where $\mu_{i,k}$ are some constants.

Indeed, from (1.55) and (1.64) it immediately follows that

$$\varphi_{i,j}(\tau) = u_{i,m-i+j+1}(\tau). \tag{1.68}$$

The last relation together with (1.67) implies

$$\begin{aligned}\Upsilon_{i,j} = &-\left(1+\frac{m^2}{(m-i+1)^2}\right)u_{j,2m-i-j+2}-\\ &-\left(1+\frac{m^2}{(m-j+1)^2}\right)u_{i,2m-i-j+2}-\\ &-m^2\left(\frac{1}{(m-i+1)^2}+\frac{1}{(m-j+1)^2}\right)u_{m,2m-i-j+2}=\\ &=\chi_{i,j}u_{m,2(m-\frac{i+j}{2}+1)}+\mathrm{Pol}\left(\{u_{m,2s}(\tau)\}_{s=1}^{m-\frac{i+j}{2}}\right),\end{aligned} \tag{1.69}$$

where $\chi_{i,j}$ are some constants. Also, from (1.62) and (1.68) it easily follows that

$$\Theta_{i,j}(\tau) =$$
$$\mathrm{Pol}\Big(\Big\{u_{l,k}(\tau) : l \in \{i,j,m\}, m-l+1 \leq k < 2m-i-j+2\Big\}\Big).$$

This together with (1.67) implies that

$$\Theta_{i,j}(\tau) = \mathrm{Pol}\Big(\{u_{m,2s}(\tau)\}_{s=1}^{m-\frac{i+j}{2}}\Big). \tag{1.70}$$

Substituting (1.69) and (1.70) into (1.60), we obtain

$$(\dot{S}_\tau^0)_{i,j} = \chi_{i,j}u_{m,2(m-\frac{i+j}{2}+1)}+\mathrm{Pol}\left(\{u_{m,2s}(\tau)\}_{s=1}^{m-\frac{i+j}{2}}\right). \tag{1.71}$$

Using (1.68), we can rewrite (1.71) in the form

$$\begin{aligned}(\dot{S}_\tau^0)_{i,j} = &\chi_{i,j}\varphi_{m,2(m-\frac{i+j}{2}+1)-1}+\\ &+\mathrm{Pol}\left(\{\varphi_{m,2s-1}(\tau)\}_{s=1}^{m-\frac{i+j}{2}}\right).\end{aligned} \tag{1.72}$$

From (1.28) and (1.31) it follows that

$$\left(\frac{1}{t-\tau}+\varphi_m(t,\tau)\right)^2 = \frac{1}{(t-\tau)^2}+\frac{1}{m^2}g(t,\tau), \tag{1.73}$$

which easily implies

$$\varphi_{m,0}(\tau) \equiv 0 \tag{1.74}$$

and

$$\varphi_{m,k}(\tau) = \frac{1}{2m^2}\beta_{0,k-1}+\mathrm{Pol}\left(\{\beta_{0,s}(\tau)\}_{s=0}^{k-2}\right), \quad k \geq 1 \tag{1.75}$$

(relation (1.74) also follows from (1.59) if we set $i = j = m$). From (1.66) and (1.74) it follows that

$$u_{i,1}(\tau) \equiv 0, \quad 1 \le i \le m. \tag{1.76}$$

Substituting (1.75) into (1.72) and using Lemma I.4.1, we finally have

$$\left(\dot{S}^0_\tau\right)_{i,j} = c_{i,j}\beta_{0,2(m-\frac{i+j}{2})}(\tau) + \operatorname{Pol}\left(\{\beta_{0,2s}(\tau)\}_{s=0}^{m-\frac{i+j}{2}-1}\right),$$

where

$$c_{i,j} = \frac{\chi_{i,j}}{2m^2}. \tag{1.77}$$

Therefore, in order to prove Lemma 1.6 it remains to prove formula (1.67). For this we refer to the proof of Theorem I.2 in Sec. I.7. Let $\alpha_{i,j}(t)$ be as in relation (7.19) from Part I. Then from item 3 of Lemma I.7.3, (1.74), and (1.68) it follows that for $i \le j \le m-1$,

$$\alpha_{j,i}(\tau) = \operatorname{Pol}\Big(\Big\{\varphi_{m,k}(\tau)\Big\}_{k=1}^{j-i}\Big) = \operatorname{Pol}\Big(\Big\{u_{m,k}(\tau)\Big\}_{k=2}^{j-i+1}\Big) \tag{1.78}$$

(actually, this implies that

$$\alpha_{i,i}(\tau) \equiv 0, \quad 1 \le i \le m-1, \tag{1.79}$$

in addition to Lemma I.7.3). Using (7.43), (7.44), and item 2 of Lemma I.7.3, it is not difficult to obtain that, similarly to (7.45) from Part I, the functions $u_{i,k}(\tau)$ satisfy the following linear system of equations:

$$\begin{aligned} \zeta_i(k)u_{i-1,k}(\tau) + \eta_i(k)u_{i,k}(\tau) + \theta_i(k)u_{m,k}(\tau) = \\ = \widetilde{\Psi}_{i,k}, \quad 1 \le i \le m, \end{aligned} \tag{1.80}$$

where

$$\begin{aligned} \zeta_i(k) &= \frac{(k+i-m-1)(i-2m-1)(i-1)}{m-i+1}, \\ \eta_i(k) &= (k+i-1)(k+i-2m-1), \\ \theta_i(k) &= \frac{k+2i-2-2m}{m-i+1}m^2, \end{aligned} \tag{1.81}$$

and

$$\begin{aligned} \widetilde{\Psi}_{i,k} = \operatorname{Pol}\left(\Big\{u_{l,n}(\tau) : l \in \{i,m\},\ 1 \le n < k\Big\}\right) + \\ + \sum_{j=i}^{m-1} \alpha_{j,i}(\tau)\operatorname{Pol}\left(\Big\{u_{s,n}(\tau) : l \in \{j,m\},\ 0 \le n < k\Big\}\right). \end{aligned} \tag{1.82}$$

Combining (1.82) with (1.76) and (1.78), one can obtain that

$$\forall k > m - i \ \widetilde{\Psi}_{i,k} = \mathrm{Pol}\left(\left\{u_{l,n}(\tau) : i \le l \le m,\ 2 \le n < k\right\}\right). \quad (1.83)$$

From identity (7.47) of Part I it follows that for nonnegative k the determinant of the matrix corresponding to the linear system (1.80) is zero iff $k \in \{2s :\ 1 \le s \le m\}$. Also, from (1.81) it easily follows that first $(m-1)$ columns of this matrix are always linearly independent. Therefore,

$$u_{i,k}(\tau) = \begin{cases} \mathrm{Lin}\Big(\big\{\widetilde{\Psi}_{j,k}(\tau)\big\}_{j=1}^{m}\Big), & k \notin \{2s : 1 \le s \le m\} \\ \mu_{i,k} u_{m,k}(\tau) + \\ \ + \mathrm{Lin}\Big(\big\{\widetilde{\Psi}_{j,k}(\tau)\big\}_{j=1}^{m}\Big), & k \in \{2s : 1 \le s \le m\}, \end{cases} \quad (1.84)$$

where by Lin(...) we denote some linear combination of the functions, contained in the brackets, and $\mu_{i,k}$ are some constants.

Note that, by (1.66), $u_{i,k}(\tau) \equiv 0$ for $1 \le i \le m-k$. This implies that for $\max\{m-k+1, 1\} \le i \le m$ we have

$$u_{i,k}(\tau) = \begin{cases} \mathrm{Lin}\Big(\big\{\widetilde{\Psi}_{j,k}(\tau)\big\}_{l=\max\{m-k+1,1\}}^{m}\Big), & k \notin \{2s : 1 \le s \le m\}, \\ \mu_{i,k} u_{m,k}(\tau) + \\ \ + \mathrm{Lin}\Big(\big\{\widetilde{\Psi}_{j,k}(\tau)\big\}_{l=\max\{m-k+1,1\}}^{m}\Big), & k \in \{2s : 1 \le s \le m\}, \end{cases} \quad (1.85)$$

Then, taking into account (1.83), we obtain

(1) for $k \notin \{2s : 1 \le s \le m\}$

$$u_{i,k}(\tau) = \\ = \mathrm{Pol}\Big(\big\{u_{l,n}(\tau) :\ \max\{m-k+1, 1\} \le l \le m,\ 2 \le n < k\big\}\Big); \quad (1.86)$$

(2) for $k \in \{2s : 1 \le s \le m\}$

$$u_{i,k}(\tau) = \mu_{i,k} u_{m,k}(\tau) + \\ + \mathrm{Pol}\Big(\big\{u_{l,n}(\tau) : \max\{m-k+1, 1\} \le l \le m,\ 2 \le n < k\big\}\Big). \quad (1.87)$$

Finally, by induction on k, starting from $k = 2$, we obtain

$$u_{i,k}(\tau) = \begin{cases} \mathrm{Pol}\left(\left\{u_{m,n}(\tau)\right\}_{n=2}^{\min\{k-1,m\}}\right), & k \notin \{2s : 1 \le s \le m\},\ k > 2 \\ \mu_{i,k} u_{m,k}(\tau) + \\ \mathrm{Pol}\left(\left\{u_{m,n}(\tau)\right\}_{n=2}^{k-1}\right), & k \in \{2s : 1 \le s \le m\}, \end{cases} \tag{1.88}$$

which implies (1.67) (see, e.g., Lemma I.4.1). The proof of Lemma 1.6 is completed. □

As a consequence of (1.67) and (1.83) we obtain that

$$\forall k > m - i \ \ \widetilde{\Psi}_{i,k} = \mathrm{Pol}\left(\left\{u_{m,2s}(\tau)\right\}_{s=1}^{\left[\frac{k-1}{2}\right]}\right). \tag{1.89}$$

Proof of Lemma 1.7. From (1.77), it follows that in order to prove that one can take $c_{i,j} \neq 0$ in (1.43), it is sufficient to show that one can take $\chi_{i,j} \neq 0$ in (1.71). For simplicity of the notation in expressions of the form

$$c u_{i,2k}(\tau) + \dots,$$

by $\dots$ we denote some functions of the type $\mathrm{Pol}\left(\left\{u_{m,2s}(\tau)\right\}_{s=1}^{k-1}\right)$.

1. Let us prove (1.44). By (1.69),

$$\begin{aligned} \Upsilon_{m,j} &= -(m^2+1)u_{j,m-j+2} - \\ &- \left(m^2 + 1 + \frac{2m^2}{(m-j+1)^2}\right) u_{m,m-j+2}. \end{aligned} \tag{1.90}$$

Apply (1.80) for $i = j$ and $k = m - j + 2$. Then

$$\begin{aligned} &\zeta_j(m-j+2)u_{j-1,m-j+2}(\tau) + \eta_j(m-j+2)u_{j,m-j+2}(\tau) + \\ &+ \theta_j(m-j+2)u_{m,m-j+2}(\tau) = \widetilde{\Psi}_{j,m-j+2}. \end{aligned} \tag{1.91}$$

From (1.59) and (1.68), it follows that

$$u_{i,2m-i-j+1} = -u_{j,2m-i-j+1}, \quad 1 \le i, j \le m.$$

In particular,

$$u_{j-1,m-j+2} = -u_{m,m-j+2}.$$

Substituting this into (1.91) and using (1.81) and (1.89), we have

$$\begin{aligned} u_{j,m-j+2} &= \frac{\xi_j(m-j+2) - \theta_j(m-j+2)}{\eta_j(m-j+2)} u_{m,m-j+2} + \dots = \\ &= -\frac{j^2 - (m^2+2m+2)j + m^3 + 2m + 1}{(m^2-1)(m-j+1)} u_{m,m-j+2} + \dots \end{aligned}$$

Substituting this into (1.90), one can easily obtain

$$\Upsilon_{m,j} = \chi_{m,j} u_{m,m-j+2} + \dots , \tag{1.92}$$

where

$$\chi_{m,j} = (m^2+1)\frac{\Big(j^2-(2m+3)j+3m+2\Big)-m^2}{(m^2-1)(m-j+1)} - \\ -\frac{2m^2}{(m-j+1)^2}. \tag{1.93}$$

We claim that $\chi_{m,j} < 0$ for $2 \le j \le m$, which implies (1.44) by (1.77). By (1.93), for this it is sufficient to prove that the polynomial $j^2-(2m+3)j+3m+2$ is negative for $2 \le j \le m$, but this can easily be done by estimating the roots of this polynomial.

2. Let us prove (1.45). By (1.69),

$$\Upsilon_{i,2} = -\left(1+\frac{m^2}{(m-i+1)^2}\right)u_{2,2m-i} - \left(1+\frac{m^2}{(m-1)^2}\right)u_{i,2m-i} - \\ -\left(\frac{m^2}{(m-i+1)^2}u_{m,m-j+2} + \frac{m^2}{(m-1)^2}\right)u_{m,2m-i}. \tag{1.94}$$

Let us express $u_{i,2m-i}$ and $u_{2,2m-i}$ in terms of $u_{m,2m-i}$ and $u_{p,n}$ with $n < 2m-i$.

We start from $u_{i,2m-i}$. Note that $\eta_i(2m-i+1)=0$. Hence, we have from (1.80)

$$\xi_i(2m-i+1)u_{i-1,2m-i+1} + \theta_i(2m-i+1)u_{m,2m-i+1} = \widetilde{\Psi}_{i,2m-i+1}$$

for $k = 2m-i+1$. Replacing i by $i+1$ and using (1.81) and (1.89) one can easily obtain that

$$u_{i,2m-i} = -\frac{\theta_{i+1}(2m-i)}{\xi_{i+1}(2m-i)}u_{m,2m-i} + \dots = \\ = \frac{m}{2m-i}u_{m,2m-i} + \dots \tag{1.95}$$

To analyze $u_{2,2m-i}$, we consider the first and second equations of system (1.80) with $k = 2m-i$. Then by a direct calculation,

$$u_{2,2m-i} = \frac{\theta_1(2m-i)\xi_2(2m-i) - \eta_1(2m-i)\theta_2(2m-i)}{\eta_1(2m-i)\eta_2(2m-i)}u_{m,2m-i} + \dots = \\ = -m\frac{(1-3m)i^2+(4m^2+3m-1)i-4m^2}{(m-1)(i-1)(2m-i+1)(2m-i)}u_{m,2m-i} + \dots \tag{1.96}$$

Substituting (1.95) and (1.96) into (1.94), one can obtain by a direct calculation that

$$\Upsilon_{i,2} = \chi_{i,2} u_{m,2m-i} + \dots, \tag{1.97}$$

where

$$\chi_{i,2} = \frac{m p_i(m)}{(2m-1)(2m-i+1)(m-i+1)(m-1)^2} \tag{1.98}$$

with

$$p_i(x) = (8i-6)x^3 - (4i^2 - 8i + 6)x^2 + (i^3 - 4i^2 + 4i - 1)x + i - 1.$$

We claim that m is not a root of the polynomial $p_i(x)$ for $2 \le i \le m-1$. Assuming the contrary, we obtain that m has to divide $i-1$ which contradicts the assumption that $i \le m-1$. Hence $\chi_{i,2} \ne 0$, which together with (1.77) implies (1.45).

3. Let us prove (1.46). By (1.69),

$$\Upsilon_{1,1} = -4u_{1,2m} - 2u_{m,2m} + \dots \tag{1.99}$$

Consider the linear system (1.80) for $k = 2m$. Note that the coefficients of the first equation of this system vanish. Taking the remaining $m-1$ equations and using the Kramer formula, one can easily obtaion

$$u_{1,2m} = (-1)^{m-1}\Bigg(\bigg(\sum_{j=2}^{m}(-1)^{j+m}\frac{\theta_j(2m)}{\xi_j(2m)}\prod_{i=2}^{j-1}\frac{\eta_i(2m)}{\xi_i(2m)}\bigg) + \\ + \prod_{i=2}^{m}\frac{\eta_i(2m)}{\xi_i(2m)}\Bigg)u_{m,2m} + \dots$$

Then from (1.81) it is not difficult to obtain

$$u_{1,2m} = C u_{m,2m} + \dots, \tag{1.100}$$

where

$$C = \left(\sum_{j=2}^{m}\frac{2m^2}{(m+j-1)(2m-j+1)}\prod_{i=2}^{j-1}\frac{(2m+i-1)(m-i+1)}{(m+i-1)(2m-i+1)}\right) + \\ + \prod_{i=2}^{m}\frac{(2m+i-1)(m-i+1)}{(m+i-1)(2m-i+1)} > 0. \tag{1.101}$$

Therefore, by (1.99),

$$\Upsilon_{1,1} = \chi_{1,1} u_{m,2m} + \dots,$$

where $\chi_{1,1} = -4\,C - 2 < 0$. This together with (1.77) implies (1.46). The proof of Lemma 1.7 and Proposition 6 is completed. $\square$

Conjecture 1. Let $\Lambda : I \mapsto L(W)$ be the curve of constant weight k. Then all items 1, 2, 3, 4, and 5 of Theorem 1 are equivalent.

Theorem 1 says that Conjecture 1 is true for regular and rank 1 curves. Note that the regular curves are the only curves such that the order of pole l in the Laurent expansion of $t \mapsto \Lambda_\tau(t)$ at $t = \tau$ is equal to 1, while the rank 1 curves of constant weight in $L(W)$ with $\dim W = m$ are the only curves of constant weight in $L(W)$ with $l = 2m - 1$ (all other curves of constant weight in $L(W)$ satisfy $l < 2m-1$). It is not difficult to show that l has to be an odd number. We also have proved a part of the Conjecture, namely, that item 4 of the Theorem 1 implies strong flatness for curves of the constant rank and weight if the order of pole l equals 3 or 5, but the proof consists of very long calculations and it is not clear yet, how to extend it for other l. Therefore, we postpone the presentation of this proof to the further publications.

2. Comparison theorems

2.1. Preliminaries. Recall that the points $t_0 \neq t_1$ are said to be conjugate for the curve $\Lambda(\cdot)$ in the Lagrange Grassmannian $L(W)$ if $\Lambda(t_0) \cap \Lambda(t_1) \neq 0$. In this case we will say also that the point t_1 is conjugate to t_0 w.r.t. the curve $\Lambda(\cdot)$. The notion of conjugate points is very important in the investigation of certain optimality properties of extremals.

The dimension of $\Lambda(t_0) \cap \Lambda(t_1)$ is called a multiplicity of the conjugate pair t_0, t_1. It is clear that if $\Lambda : I \mapsto L(W)$ is a nondecreasing ample curve defined on some compact interval $I \subset \mathbb{R}$, then every $t \in I$ is conjugate to a finite number of points. The natural and important problem is to estimate the number of points conjugate to the given point on the given interval in terms of symplectic invariants of the curve. In order to obtain such estimates, we will use two distinct approaches, described by Theorems 2 and 3 below. We will apply both approaches in more details to the rank 1 curves of constant weight in $L(W)$ with $\dim W = 4$.

The first approach uses the properties of the derivative curve and gives the following simple sufficient condition for absense of the points conjugate to the given point.

Theorem 2. *Let $\Lambda : I \mapsto L(W)$ be an ample nondecreasing curve such that its derivative curve $\Lambda^0(t)$ is nonincreasing. Then for any point $\tau_0 \in I$ there is no point $\tau \in I$ conjugate to τ_0.*

Proof. The method of the proof is a direct generalization of the method used in the proof of the theorem from [2] on p. 374. Obviously, it is sufficient to prove that for a given $\tau_0 \in I$ there is no $\tau > \tau_0$ conjugate to τ_0. One can introduce coordinates in W, $W = \mathbb{R}^m \oplus \mathbb{R}^m = \{(x, y) : x, y \in \mathbb{R}^m\}$, such that the symplectic form σ has a standard form,

$$\sigma((x_1, y_1), (x_2, y_2)) = \langle x_2, y_1\rangle - \langle x_1, y_2\rangle, \tag{2.1}$$

the subspace $\Lambda(\tau_0)$ satisfies $\Lambda(\tau_0) = \mathbb{R}^m \oplus 0$ and $\Lambda_0(\tau_0) = \{(x, S^0 x) : x \in \mathbb{R}^m\}$, where S^0 is a positive definite symmetric matrix (for brevity we will write $S^0 > 0$). Let $\Delta = 0 \oplus \mathbb{R}^m$. Let S_τ, S^0_τ be the matrices corresponding to $\Lambda(\tau)$ and $\Lambda^0(\tau)$ in the chosen coordinates. Note that, in general, the matrix S_τ (S^0_τ) is defined if the subspace $\Lambda(\tau)$ ($\Lambda^0(\tau)$) belongs to $\Delta^\pitchfork$. By construction, $S_{\tau_0} = 0$, $S^0_{\tau_0} = S^0 > 0$, i.e.,

$$S^0_{\tau_0} - S_{\tau_0} > 0. \tag{2.2}$$

Recall that from the construction of the derivative curve it follows that $\Lambda(\tau) \cap \Lambda^0(\tau) = 0$ for any τ. Therefore,

$$\det(S^0_\tau - S_\tau) \neq 0. \tag{2.3}$$

Further, by the assumptions,

$$\dot S_\tau \geq 0 \quad \dot S^0_\tau \leq 0. \tag{2.4}$$

Relations (2.3) and (2.4) hold until $\Lambda(\tau)$ and $\Lambda^0(\tau)$ remain in $\Delta^\pitchfork$. However, from (2.2)–(2.4) and the assumption that the curve $\Lambda(\cdot)$ is ample it follows that

$$0 < S_\tau \leq S^0_\tau \leq S^0_{\tau_0}.$$

Hence $\Lambda(\tau)$ and $\Lambda(\tau_0)$ do not leave the set $\Delta^\pitchfork$ at all. This completes the proof of the theorem. □

The second approach to the estimation of conjugate points is based on the following, simplified for our purposes, version of the multidimensional generalization of the classical Sturm theorems about zeros of solutions of second order differential equations (for more general formulation and proof see [4] and [5]).

Theorem 3. *Let h_τ and H_τ be quadratic nonstationary Hamiltonians on W such that for any $0 \leq \tau \leq t$ the quadratic form $H_\tau - h_\tau$ is nonnegative definite. Let P_τ, $\tilde P_\tau \in \mathrm{Sp}(W)$ be linear Hamiltonian flows on W, generated by the Hamiltonian fields $\vec h_\tau$ and $\vec H_\tau$:*

$$\frac{\partial}{\partial\tau} P_\tau = \vec h_\tau P_\tau, \quad \frac{\partial}{\partial\tau} \tilde P_\tau = \vec H_\tau \tilde P_\tau, \quad P_0 = \tilde P_0 = \mathrm{id}\,.$$

Finally, let $\Lambda(\tau)$ and $\tilde\Lambda(\tau)$ be nondecreasing ample trajectories of the corresponding flows on $L(W)$:

$$\Lambda(\tau) = P_\tau \Lambda(0), \quad \tilde\Lambda(\tau) = \tilde P_\tau \tilde\Lambda(0), \quad 0 \leq \tau \leq t.$$

Then, for any $\Lambda_1 \in L$, which is transversal to the endpoints of the curves $\Lambda(\cdot)$ and $\tilde{\Lambda}(\cdot)$, the inequality

$$\sum_{0\leq\tau\leq t} \dim\left(\Lambda(\tau)\cap\Lambda_1\right) - \frac{1}{2}\dim W \leq \sum_{0\leq\tau\leq t} \dim\left(\tilde{\Lambda}(\tau)\cap\Lambda_1\right) \tag{2.5}$$

is valid.

Here, as in the introduction to [1], the Hamitonian field $\vec{h}$ corresponding to the Hamiltonian h is defined by the identity $\sigma(\vec{h},\cdot) = dh(\cdot)$.

Assume that $\Lambda : \mathbb{R} \mapsto L(W)$ is a nondecreasing curve of rank 1 and constant weight. Let again $\big(e_1(\tau,\dots,e_m(\tau),f_1(\tau),\dots,f_m(\tau)\big)$ be the canonical moving frame associated with the curve $\Lambda(\tau)$. As in the proof of Theorem I.2, denote by $E(\tau)$ and $F(\tau)$ the tuples of vectors $(e_1(\tau),\dots,e_m(\tau))$ and $(f_1(\tau),\dots,f_m(\tau))$ respectively, arranged in the columns. Let us fix the bases $E(\tau)$ and $F(\tau)$ in $\Lambda(\tau)$ and $\Lambda^0(\tau)$ respectively, let S_t be the matrix, corresponding to the linear mapping $\langle\Lambda(\tau),\Lambda(t),\Lambda^0(\tau)\rangle$, and let S_t^0 be the matrix, corresponding to the linear mapping $\langle\Lambda^0(\tau),\Lambda^0(t),\Lambda(\tau)\rangle$ (see Sec. I.2 for notation).

First note that, by (1.47), Theorem 2 can be formulated in the following form.

Corollary 4. *Let $\Lambda : I \mapsto L(W)$ be a nondecreasing rank 1 curve of constant weight such that for any $\tau \in I$ the matrix $\dot{S}^0_\tau$ is nonnegative definite. Then for any point $\tau_0 \in I$ there is no point $\tau \in I$ conjugate to τ_0.*

Hence, if one finds an explicit expression for matrix $\dot{S}^0_\tau$ in terms of the complete system of invariants of the curve $\Lambda(\tau)$ (existence of such expression follows from the proof of Theorem I.2), then, by Corollary 4, one can obtain explicit sufficient conditions for absence of conjugate points in terms of the complete system of invariants of the curve $\Lambda(\tau)$.

Now we will explain how to estimate the number of conjugate points, for example, to the point 0 w.r.t. the curve $\Lambda(\cdot)$, using Theorem 3. It is easy to see that the structural equation for the canonical moving frame has the following form:

$$\begin{pmatrix}\dot{E}(\tau)\\ \dot{F}(\tau)\end{pmatrix} = \begin{pmatrix}\Omega(\tau) & \dot{S}_\tau\\ \dot{S}^0_\tau & -\Omega^T(\tau)\end{pmatrix}\begin{pmatrix}E(\tau)\\ F(\tau)\end{pmatrix} \tag{2.6}$$

(here $\Omega(\tau)$ is an $m\times m$ matrix with the (i,j)th entry equal to $\alpha_{i,j}(\tau)$, where $\alpha_{i,j}(\tau)$ is defined by (7.19) from Part I with $\Delta = \Lambda^0(\tau)$). Denote by $\mathcal{B}(\tau)$ the matrix in the structural equation (2.6), namely,

$$\mathcal{B}_\tau = \begin{pmatrix}\Omega(\tau) & \dot{S}_\tau\\ \dot{S}^0_\tau & -\Omega^T(\tau)\end{pmatrix}.$$

Fixing the basis $e_1(0), \dots, e_m(0), f_1(0), \dots, f_m(0)$ in W, we identify W with $\mathbb{R}^m \oplus \mathbb{R}^m = \{(x,y) : x, y \in \mathbb{R}^m\}$ such that $\Lambda^0(0) = 0 \oplus \mathbb{R}^m$ and the symplectic form σ has the standard form (2.1). Denote by $\mathcal{F}_\tau$ the $2m \times 2m$ matrix such that its ith column is equal to the coordinates of $e_i(\tau)$ for $1 \le i \le m$ and to the coordinates of $f_i(\tau)$ for $m+1 \le i \le 2m$ (the coordinates are w.r.t. the chosen basis $e_1(0), \dots, e_m(0), f_1(0), \dots, f_m(0)$). One can look at $\mathcal{F}_\tau$ at on the linear flow on W (which in turn, generates the flow on $L(W)$). Then, by construction, $\Lambda(\tau) = \mathcal{F}_\tau \Lambda(0)$, and from the structural equation (2.6) it follows that

$$\frac{d}{d\tau}\mathcal{F}_\tau = \mathcal{F}_\tau \mathcal{B}_\tau^T, \quad \mathcal{F}_0 = \mathrm{id}\,.$$

Denote $P_\tau = \mathcal{F}_{-\tau}^{-1}$ and let h_τ^1 be the following nonstationary Hamiltonian:

$$h_\tau^1((x,y)) = \frac{1}{2}\Big(\langle \dot S_{-\tau} x, x\rangle - 2\langle \Omega(-\tau)^T x, y\rangle - \langle \dot S_{-\tau}^0 y, y\rangle\Big). \tag{2.7}$$

Then it is easy to show that P_τ is a linear Hamiltonian flow on W, generated by the Hamiltonian vector field $\vec h_\tau^1$:

$$\frac{\partial}{\partial\tau} P_\tau = \mathcal{B}_{-\tau}^T P_\tau = \vec h_\tau^1 P_\tau, \quad P_0 = \mathrm{id}\,. \tag{2.8}$$

Moreover, if $\Upsilon(\cdot)$ is the trajectory of the flow P_τ, starting from $\Lambda(0)$, i.e.,

$$\Upsilon(\tau) = P_\tau \Lambda(0), \tag{2.9}$$

then

$$\begin{aligned}\dim(\Upsilon(\tau)\cap\Upsilon(0)) &= \dim(\mathcal{F}_{-\tau}^{-1}\Lambda(0)\cap\Lambda(0)) = \\ &= \dim(\Lambda(0)\cap\mathcal{F}_{-\tau}\Lambda(0) = \dim(\Lambda(0)\cap\Lambda(-\tau)).\end{aligned} \tag{2.10}$$

This yields that the point τ_1 is conjugate to the point 0 w.r.t. the curve $\Upsilon(\cdot)$ iff the point $-\tau_1$ is conjugate to 0 w.r.t. the original curve $\Lambda(\cdot)$. Also, we note that by construction, the curve $\Upsilon(\cdot)$ is a nondecreasing ample curve. Therefore, we can deal with the curve $\Upsilon(\cdot)$ instead of $\Lambda(\cdot)$.

The scheme for the estimation of the numbers of conjugate points is as follows: first, one can express the structural equation (2.6) (and, therefore, also the Hamiltonian h_τ^1) in terms of the complete system of invariants of the curve. Then one can compare h_τ^1 with some simple (for example, stationary) Hamiltonian h^0 on W such that for the trajectories of the corresponding linear Hamiltonian field $\vec h^0$ the pairs of conjugate points can be computed explicitly. Finally, one can use an inequality of type (2.5) to obtain the estimates for the number of conjugate points for the original curve.

In the both approaches we have to know the expression for the structural equation (2.6) in terms of the complete system of invariants of the curve.

The problem is that in general such expression is rather complicated. Therefore, from now we restrict ourselves to curves of rank 1 and constant weight for $m = 2$, i.e., $\dim W = 4$.

2.2. Structural equation in the case $r = 1$, $m = 2$. In this case the complete system of invariants consists of the pair of functions $(\rho(t), \beta_{0,2}(t))$ or $(\rho(t), A(t))$, where $A(t)$ is the density of the fundamental form $\mathcal{A}$ (see also (5.8) from Part I).

Proposition 7. *The canonical moving frame $\Big(e_1(\tau), e_2(\tau), f_1(\tau), f_2(\tau)\Big)$ of the curve $\Lambda(\cdot)$ satisfies the following structural equation:*

$$\begin{pmatrix} \dot e_1(\tau) \\ \dot e_2(\tau) \\ \dot f_1(\tau) \\ \dot f_2(\tau) \end{pmatrix} =$$

$$= \begin{pmatrix} 0 & 3 & 0 & 0 \\ \frac{1}{4}\rho(\tau) & 0 & 0 & 4 \\ -\left(\frac{35}{36}A(\tau) - \frac{1}{8}\rho(\tau)^2 + \frac{1}{16}\rho''(\tau)\right) & -\frac{7}{16}\rho'(\tau) & 0 & -\frac{1}{4}\rho(\tau) \\ -\frac{7}{16}\rho'(\tau) & -\frac{9}{4}\rho(\tau) & -3 & 0 \end{pmatrix} \times$$

$$\times \begin{pmatrix} e_1(\tau) \\ e_2(\tau) \\ f_1(\tau) \\ f_2(\tau) \end{pmatrix}. \tag{2.11}$$

As a direct consequence of Corollary 4 and Proposition 7, we obtain the following theorem.

Theorem 4. *Let $\Lambda : I \mapsto L(W)$ be a rank 1 curve of constant weight with the Ricci curvature $\rho(\cdot)$ and density $A(\cdot)$ of the fundamental form. If the following symmetric matrix*

$$\begin{pmatrix} -\left(\frac{35}{36}A(\tau) - \frac{1}{8}\rho(\tau)^2 + \frac{1}{16}\rho''(\tau)\right) & -\frac{7}{16}\rho'(\tau) \\ -\frac{7}{16}\rho'(\tau) & -\frac{9}{4}\rho(\tau) \end{pmatrix}$$

is nonnegative definite, then for $\tau_0 \in I$ there is no point $\tau \in I$ conjugate to τ_0.

Sketch of the proof of Proposition 7. Here we give the main steps of the computation leaving the details to the reader. We use the notations of (2.6) with $\Omega(\tau) = (\alpha_{ij}(\tau))_{1\le i,j\le 2}$. First, from item 2 of Lemma I.7.3 it follows that $\alpha_{1,2} = 3$. Using identity (1.31) and the fact that $\beta_{0,2}(\tau) = \frac{1}{2}\rho'(\tau)$ (see (4.18) of Part I), it is easy to obtain the following expansion for $w(t,\tau)$:

$$w_2(t,\tau) = \frac{1}{(t-\tau)} + \frac{1}{8}\rho(\tau)(t-\tau) + \frac{1}{16}\rho'(\tau)(t-\tau)^2 + \\ + a_2(\tau)(t-\tau)^3 + O((t-\tau)^4), \tag{2.12}$$

where

$$a_2(\tau) = \frac{1}{8}\beta_{0,2}(\tau) - \frac{1}{128}\rho(\tau)^2. \tag{2.13}$$

This implies, in particular, that $\varphi_2(\tau,\tau) = 0$, where the function $\varphi_2(t,\tau)$ is as in (1.28). Therefore, the function $\Phi_2(t,\tau)$, defined by (7.26) from Part I, satisfies also $\Phi_2(\tau,\tau) = 0$ (or, in the notation of (7.31) from Part I, $c_0(\tau) = 0$). Then, by (7.38) of Part I, we have

$$\alpha_{1,1}(\tau) = c_0(\tau) + 2\varphi_2(\tau,\tau) + \frac{\partial}{\partial t}\varphi_2(t,\tau)\Big|_{t=\tau} = 0. \tag{2.14}$$

Further, by Eq. (7.39) of Part I, we have for $m = i = 2$

$$\frac{\partial Y_1}{\partial t} = -\frac{1}{3}\left(\frac{\partial^2}{\partial t\partial\tau}(\ln w_2) + 4w_2^2\right), \tag{2.15}$$

where $Y_1 = \dfrac{w_1}{w_2}$ (to obtain (2.15) from (7.38) of Part I, we have used that $Y_2 \equiv 1$). Then, by a direct computation, one can obtain from (2.12) the following expansion for $\dfrac{\partial Y_1}{\partial t}$:

$$\frac{\partial Y_1}{\partial t} = -\frac{1}{(t-\tau)^2} - \frac{1}{4}\rho(\tau) - \frac{1}{8}\rho'(\tau)(t-\tau) - \\ - \left(\frac{1}{16}\rho''(\tau) - \frac{4}{3}a_2(\tau) + \frac{5}{96}\rho(\tau)^2\right)(t-\tau)^2 + \\ + O((t-\tau)^3). \tag{2.16}$$

Hence

$$w_1(t,\tau) = Y_1(t,\tau)w_2(t,\tau) = \Bigg(\frac{1}{t-\tau} + C - \frac{1}{4}\rho(\tau)(t-\tau) - \\ - \frac{1}{16}\rho'(\tau)(t-\tau)^2 - \frac{1}{3}\left(\frac{1}{16}\rho''(\tau) - \frac{4}{3}a_2(\tau) + \frac{5}{96}\rho(\tau)^2\right)(t-\tau)^2 + \\ + O((t-\tau)^4)\Bigg)\Bigg(\frac{1}{(t-\tau)} + \frac{1}{8}\rho(\tau)(t-\tau) + \frac{1}{16}\rho'(\tau)(t-\tau)^2 + \\ + a_2(\tau)(t-\tau)^3 + O((t-\tau)^4)\Bigg). \tag{2.17}$$

By our construction, the coefficient of $\frac{1}{t-\tau}$ in the expansion of $t \to w_1(t,\tau)$ into the Laurent series at $t = \tau$ vanishes. Opening brackets in (2.17), we find that the constant C is equal to zero. Moreover, the expansion of w_1 has the following form:

$$w_1(t,\tau) = \frac{1}{(t-\tau)^2} - \frac{1}{8}\rho(\tau) + a_1(\tau)(t-\tau)^2, \tag{2.18}$$

where

$$a_1(\tau) = \frac{13}{9}a_2(\tau) - \frac{1}{48}\rho''(\tau) - \frac{7}{144}\rho(\tau)^2. \tag{2.19}$$

Remark 6. Note that the asymptotic of w_1 up to $O((t-\tau)^2)$ can be obtained easier, using the fact that the expansions of $t \mapsto w_1(t,\tau)w_i(t,\tau)$, $i = 1, 2$, into the Laurent series at $t = \tau$ do not contain the terms of the type $\frac{c}{t-\tau}$ (otherwise the expansions of $\big((S_t - S_\tau)^{-1}\big)\big)_{1i} = \int_\tau^t w_1(\xi,\tau)w_i(\xi,\tau)d\xi$ contain logarithmic terms).

To find $\alpha_{2,1}(\tau)$ and $\alpha_{2,2}(\tau)$, we use relation (I.7.52) from Part I and the sentence after it: the function $\alpha_{2,i}(\tau)$ is equal to the coefficient of $\frac{1}{t-\tau}$ in the expansion of

$$t \mapsto -4w_2(t,\tau)\int^t w_2(\xi,\tau)w_i(\xi,\tau)\,d\xi$$

into the Laurent series at $t = \tau$. This coefficient can easily be obtained from (2.12) and (2.18), namely,

$$\alpha_{2,1}(\tau) = \frac{1}{4}\rho(\tau), \quad \alpha_{2,2} = 0. \tag{2.20}$$

By (2.6) it remains to find the matrix $\dot{S}^0_\tau$. Using relation (7.50) from Part I and expansions (2.12) and (2.18), the reader will have no difficulty to show that

$$\dot{S}^0_\tau = \begin{pmatrix} -4a_2(\tau) + \frac{1}{64}\rho(\tau)^2 - 2a_1(\tau) & -\frac{7}{16}\rho'(\tau) \\ -\frac{7}{16}\rho'(\tau) & -\frac{9}{4}\rho(\tau) \end{pmatrix}. \tag{2.21}$$

Finally, combining formulas (5.8) of Part I, (2.13), and (2.19), one can easily obtain that

$$\left(\dot{S}^0_\tau\right)_{1,1} = -\left(\frac{35}{36}A(\tau) - \frac{1}{8}\rho(\tau)^2 + \frac{1}{16}\rho''(\tau)\right). \tag{2.22}$$

This completes the proof of the proposition. □

Remark 7. From (2.11) it can be shown by a direct calculation that the curvature operator $R(\tau) : \Lambda(\tau) \to \Lambda(\tau)$ of the curve $\Lambda(\cdot)$ of rank 1 and constant weight has the following matrix in the basis $(e_1(\tau), e_2(\tau))$:

$$R(\tau) = -\dot{S}^0_\tau \, \dot{S}_\tau = \begin{pmatrix} 0 & \frac{7}{4}\rho'(\tau) \\ 0 & 9\rho(\tau) \end{pmatrix}, \tag{2.23}$$

i.e., $R(\tau)$ depends only on $\rho(\tau)$. From this and Corollary 1 it follows that in contrast to the regular curves (see Lemma 1.2), for rank 1 curves the fact that the curvature operator of the curve is identically equal to zero does not imply that the curve is flat.

As you see from (2.11), even for $m = 2$ the structural equation is complicated, for example, it contains the derivatives of $\rho(t)$ up to the second order. It is not so clear with what simple Hamiltonian system it can be compared. Therefore, to apply Theorem 3, first we obtain the comparison theorems in the case where $\rho(\tau) \equiv 0$, i.e., where τ is a projective parameter. To obtain the comparison theorems for an arbitrary parameter, we will make reparametrization to a projective parameter. The invariant $A(t)$ is more convenient than $\beta_{0,2}$, since it has a simpler reparametrization rule.

2.3. The case of the projective parameter. First suppose that the Ricci curvature $\rho(\tau)$ of the curve $\Lambda(\tau)$ is identically equal to zero, i.e., τ is a projective parameter. Then structural equation (2.11) has the following simple form:

$$\begin{pmatrix} \dot{e}_1(\tau) \\ \dot{e}_2(\tau) \\ \dot{f}_1(\tau) \\ \dot{f}_2(\tau) \end{pmatrix} = \begin{pmatrix} 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 4 \\ -\frac{35}{36}A(\tau) & 0 & 0 & 0 \\ 0 & 0 & -3 & 0 \end{pmatrix} \begin{pmatrix} e_1(\tau) \\ e_2(\tau) \\ f_1(\tau) \\ f_2(\tau) \end{pmatrix}. \tag{2.24}$$

In this case the Hamiltonian h^1_τ, defined by (2.7), has the form

$$h^1_\tau(x, y) = \frac{1}{2}\left(\frac{35}{36}A(-\tau)y_1^2 - 6x_1y_2 + 4x_2^2\right), \tag{2.25}$$

where $x = (x_1, x_2)$ and $y = (y_1, y_2)$.

We will compare this Hamiltonian with the following stationary Hamiltonian h^{A_0}:

$$h^{A_0}(x, y) = \frac{1}{2}\left(\frac{A_0}{36}y_1^2 - 6x_1y_2 + 4x_2^2\right) \tag{2.26}$$

for some constant A_0. The corresponding Hamiltonian system has the form

$$\begin{cases} \dot{x}_1 = -\dfrac{A_0}{36} y_1, \\ \dot{x}_2 = 3x_1, \\ \dot{y}_1 = -3y_2, \\ \dot{y}_2 = 4x_2. \end{cases} \tag{2.27}$$

Let $\Gamma(\tau)$ be the trajectory of the flow generated on $L(W)$ by the system (2.27) such that $\Gamma(0) = \Lambda(0)$. It is easy to see that $\Gamma(\tau)$ is ample non-decreasing curve. By a direct computation one can obtain the following lemma.

Lemma 2.1. *If $A_0 \leq 0$, then there are no points $\tau \neq 0$ conjugate to 0 w.r.t. the curve $\Gamma(\tau)$. If $A_0 > 0$, then the point τ_1 is conjugate to 0 w.r.t. $\Gamma(\tau)$ iff τ_1 is a solution of the equation*

$$\cos(\omega\tau)\cosh(\omega\tau) = 1, \tag{2.28}$$

where $\omega = \sqrt[4]{A_0}$. In addition, if $\tau_1 \neq 0$ is conjugate to 0, then the multiplicity of the conjugate pair (τ_0, τ_1) is equal to 1.

Denote by p_1 the first positive solution of the equation

$$\cos\tau \cosh\tau = 1. \tag{2.29}$$

Note that $p_1 \approx 4.73004$. Using Theorem 3 and Lemma 2.1, one can obtain the following theorem

Theorem 5 (Comparison theorem in the projective parameter). *Let $\Lambda(\tau)$ be a curve of rank 1 and constant weight in $L(W)$ defined on the interval $I \subseteq \mathbb{R}$ such that its Ricci curvature $\rho(\tau)$ is identically equal to zero on I (in other words τ is a projective parameter). Then the following three statements are valid:*

1. *if $A(\tau) \leq 0$, then for any point $\tau_0 \in I$ there are no points $\tau \neq \tau_0$ conjugate to τ_0;*
2. *if $A(\tau) \leq \dfrac{A_0}{35}$ for some constant $A_0 > 0$, then for any $\tau_0 \in I$ there are no points conjugate to τ_0 in the interval $\left(\tau_0, \tau_0 + \dfrac{p_1}{\sqrt[4]{A_0}}\right)$;*
3. *if $A(\tau) \geq \dfrac{A_0}{35}$ for some constant $A_0 > 0$, then for any pair of consequent conjugate points τ_0, τ_1 in I the inequality $|\tau_1 - \tau_0| \leq \dfrac{p_1}{\sqrt[4]{A_0}}$ holds.*

Proof. First we can suppose that $I = \mathbb{R}$ (otherwise, we can extend somehow the function $A(\tau)$ to the whole $\mathbb{R}$, preserving its upper bound (or lower bound), and work with the Hamiltonian h^1_τ defined by (2.25) for all $\tau \in \mathbb{R}$). Without loss of generality, one can also suppose that $\tau_0 = 0$ (otherwise, one can work with the curve $\tau \mapsto \Lambda(\tau + \tau_0)$ instead of $\Lambda(\tau)$). As before, let $\Gamma(\cdot)$ and $\Upsilon(\cdot)$ be the trajectories of the flows generated on $L(W)$ by the Hamiltonian vector fields $\vec{h}^{A_0}$ and h^1_τ, respectively, such that $\Gamma(0) = \Upsilon(0) = \Lambda(0)$.

First we prove items 1 and 2 of the theorem. Suppose that $A(\tau) \le \frac{A_0}{35}$. For some numbers ϵ and c such that $c > \epsilon > 0$, we consider two curves

$$\tilde{\Gamma}(\tau) = \Gamma(\tau - c), \quad \tilde{\Upsilon}(\tau) = \Upsilon(\tau - c + 2\epsilon)$$

on the interval $0 \le \tau \le c - \epsilon$. The first curve is again a trajectory of the flow generated on $L(W)$ by the stationary Hamiltonian vector field $\vec{h}^{A_0}$. The second curve is a trajectory of the flow generated on $L(W)$ by the nonstationary Hamiltonian vector field $\vec{h}^1_{(\tau - c + 2\epsilon)}$. By assumptions and (2.25) and (2.26), the quadratic form $h^{A_0} - h^1_{(\tau - c + 2\epsilon)}$ is nonnegative definite for any τ. In the case $A_0 > 0$ suppose also that $c \le \frac{p_1}{\sqrt[4]{A_0}}$. Then in both cases by the previous lemma the curve $\tilde{\Gamma}(\tau)$, $0 \le \tau \le c - \epsilon$, belongs to $\Lambda(0)^{\pitchfork}$. Obviously, the subspaces $\tilde{\Upsilon}(0)$ and $\tilde{\Upsilon}(c - \epsilon)$ are also transversal to $\Lambda(0)$ for sufficiently small $\epsilon > 0$. Therefore, one can apply Theorem 3 to the curves $\tilde{\Gamma}(\cdot)$ and $\tilde{\Upsilon}(\cdot)$ on the interval $[0, c - \epsilon]$:

$$\sum_{0 \le \tau \le c - \epsilon} \tilde{\Upsilon}(\tau) \cap \Lambda(0) - 2 \ \le \sum_{0 \le \tau \le c - \epsilon} \tilde{\Gamma}(\tau) \cap \Lambda(0) = 0. \tag{2.30}$$

On the other hand, by definition $\tilde{\Upsilon}(c - 2\epsilon) = \Upsilon(0) = \Lambda(0)$. Therefore,

$$\sum_{0 \le \tau \le c - \epsilon} \tilde{\Upsilon}(\tau) \cap \Lambda(0) \ge 2.$$

This together with (2.30) implies that $\sum_{0 \le \tau \le c - \epsilon} \tilde{\Upsilon}(\tau) \cap \Lambda(0) = 2$. But this means that for any $\tau \ne 0$ such that $2\epsilon - c \le \tau \le \epsilon$ we have $\Upsilon(\tau) \cap \Lambda(0) = 0$. By (2.10) this is equivalent to the fact that for any $\tau \ne 0$ such that $-\epsilon \le \tau \le c - 2\epsilon$ we have $\Lambda(\tau) \cap \Lambda(0) = 0$. Since $\epsilon > 0$ is arbitraril small, we can conclude that there are no points conjugate to 0 w.r.t. the curve $\Lambda(\cdot)$ in the interval $(0, c)$. Let us recall that in the case $A_0 > 0$ we can take $0 \le c \le \frac{p_1}{\sqrt[4]{A_0}}$. This completes the proof of item 2 of the theorem. In the case $A_0 \le 0$ any positive number can be taken as c. Hence any positive τ is not conjugate to 0 w.r.t. $\Lambda(\cdot)$. By the arguments given in the beginning of the proof, we obtain that for any τ_0, all $\tau > \tau_0$ are not conjugate to τ_0

w.r.t. $\Lambda(\cdot)$. But then also all $\tau < \tau_0$ are not conjugate to τ_0 w.r.t. $\Lambda(\cdot)$, since the notion of the conjugate points is symmetric. This completes the proof of item 1 of the theorem.

Now we prove third item of the theorem. Suppose that $A(\tau) \geq \dfrac{A_0}{35} > 0$. Again for some numbers ϵ and c such that $c > \epsilon > 0$ consider two curves

$$\bar{\Gamma}(\tau) = \Gamma(\tau - c + 2\epsilon), \quad \bar{\Upsilon}(\tau) = \Upsilon(\tau - c)$$

on the interval $0 \leq \tau \leq c - \epsilon$. The first curve is again a trajectory of the flow generated on $L(W)$ by the stationary Hamiltonian vector field $\vec{h}^{A_0}$. The second curve is a trajectory of the flow generated on $L(W)$ by the nonstationary Hamiltonian vector field $\vec{h}^1_{(\tau-c)}$. By assumptions and (2.25) and (2.26), the quadratic form $h^1_{(\tau-c)} - h^{A_0}$ is nonnegative definite for any τ. Suppose also that $c - 2\epsilon > \dfrac{p_1}{\sqrt[4]{A_0}}$ and $\epsilon < \dfrac{p_1}{\sqrt[4]{A_0}}$. Note that $\Gamma(0) = \Lambda(0)$. From the previous lemma there exists at least one point conjugate to 0 w.r.t. the curve $\Gamma(\cdot)$ in the interval $(2\epsilon - c, 0)$. Therefore, we have

$$\sum_{0 \leq \tau \leq c-\epsilon} \bar{\Gamma}(\tau) \cap \Lambda(0) = \sum_{2\epsilon - c \leq \tau \leq \epsilon} \Gamma(\tau) \cap \Lambda(0) \geq 3. \tag{2.31}$$

Also, we note that the subspaces $\bar{\Upsilon}(0)$ and $\bar{\Upsilon}(c-\epsilon)$ are transversal to $\Lambda(0)$ for sufficiently small $\epsilon > 0$. Hence, using Theorem 3 and (2.31), we obtain the following inequality:

$$\begin{aligned} \sum_{-c \leq \tau \leq -\epsilon} \Upsilon(\tau) \cap \Lambda(0) = \sum_{0 \leq \tau \leq c-\epsilon} \bar{\Upsilon}(\tau) \cap \Lambda(0) \geq \\ \geq \sum_{0 \leq \tau \leq c-\epsilon} \bar{\Gamma}(\tau) \cap \Lambda(0) - 2 \geq 1. \end{aligned} \tag{2.32}$$

This together with (2.10) implies that there is at least one point conjugate to 0 w.r.t. the curve $\Lambda(\cdot)$ in the interval (ϵ, c). Taking into account that $\epsilon > 0$ is arbitraril small and $c > \dfrac{p_1}{\sqrt[4]{A_0}} + 2\epsilon$, we can conclude that there is at least one point conjugate to 0 w.r.t. the curve $\Lambda(\cdot)$ in the interval $\left(0, \dfrac{p_1}{\sqrt[4]{A_0}}\right]$. The proof of the third part of the theorem is completed. □

2.4. The case of an arbitrary parameter. Now we suppose that the Ricci curvature $\rho(t)$ of the curve $\Lambda(t)$ is not identically equal to zero. The method proposed here is in essence a reduction to the previous case by making reparametrization $\tau = \varphi(t)$ to some projective parameter τ. From (5.4) of Part I it follows that $\tau = \varphi(t)$ is a reparametrization to a projective parameter iff the function $\varphi(t)$ satisfies the equation

$$\mathbb{S}\left(\varphi(t)\right) = \frac{3\rho(t)}{4}, \tag{2.33}$$

where $\mathbb{S}\left(\varphi(t)\right)$ is the Schwartzian of the function $\varphi(t)$ and $\rho(t)$ is the Ricci curvature (note that in the considered case the weight k of the curve $\Lambda(t)$ is equal to 4). By a direct calculation we can obtain the following lemma that will be also useful in the sequel.

Lemma 2.2. *The monotone increasing function $\varphi(t)$ satisfies Eq.* (2.33) *iff the function $y(t) = \dfrac{1}{\sqrt{\varphi'(t)}}$ satisfies the Hill equation*

$$y'' + \frac{3\rho(t)}{4} y = 0. \tag{2.34}$$

Also, let us recall that the density $\bar{A}(\tau)$ of the fundamental form in the new parameter τ satisfies

$$\bar{A}(\tau) = \frac{A(t)}{\varphi'(t)^4}. \tag{2.35}$$

Using (2.35), Lemma 2.2, and Theorem 5 we obtain a series of comparison theorems according to the different bounds of $\rho(t)$ and $A(t)$. We consider separately three cases corresponding to the three items of Theorem 5.

2.4.1. *The case $A(t) \le 0$.* In this case we have the following theorem.

Theorem 6. *Let $\Lambda : I \mapsto L(W)$ be a curve of rank* 1 *and constant weight defined on the interval $I \subseteq \mathbb{R}$ such that $A(t) \le 0$ for all $t \in I$. Then the following statements are valid*:

(1) *if $\rho(t) \le 0$, then for any point t_0 there are no points $t \neq t_0$ conjugate to t_0*;
(2) *if $\rho(t) \le \frac{4}{3}R$ for some constant $R > 0$, then for any t_0 there are no points conjugate to t_0 in the interval $\left(t_0, t_0 + \dfrac{\pi}{\sqrt{R}}\right)$*.

Proof. From the classical Sturm comparison theorem it follows that Eq. (2.34) has positive solutions on the whole $\mathbb{R}$ if $\rho(t) \le 0$ or on the interval $\left[t_0, t_0 + \dfrac{\pi}{\sqrt{R}}\right)$ for any given t_0 if $\rho(t) \le \frac{4}{3}R$ with $R > 0$. Then, by Lemma 2.2, Eq. (2.33) has a monotone increasing solution $\varphi(t)$ on the whole $\mathbb{R}$ in the first case and on the interval $\left[t_0, t_0 + \dfrac{\pi}{\sqrt{R}}\right)$ in the second case. Let $\bar{\Lambda}(\tau) = \Lambda(\varphi^{-1}(\tau))$, $\tau_0 = \varphi(t_0)$. By construction, the curve $\bar{\Lambda}(\tau)$ has vanishing Ricci curvature. By assumptions and (2.35), the corresponding density

$\bar{A}(\tau) \leq 0$. Then by Theorem 5 there are no points $\tau \neq \tau_0$ conjugate to τ_0 w.r.t. the curve $\bar{\Lambda}(\tau)$. But this is equivalent to the statement of the theorem. □

2.4.2. *The case* $A(t) \leq \frac{A_0}{35}$, $A_0 > 0$. First we prove the following auxiliary lemma.

Lemma 2.3. *Let* $\tau = \varphi(t)$ *be a reparametrization of the curve* $\Lambda(t)$ *to the projective parameter* τ *such that the function* $\varphi(t)$ *is monotone increasing. Assume that* $A(t) \leq \frac{A_0}{35}$ *and the inequality*

$$\frac{\varphi(t_0+s)-\varphi(t)}{\min\limits_{t_0 \leq t \leq t_0+s} \varphi'(t)} \leq \frac{p_1}{\sqrt[4]{A_0}} \tag{2.36}$$

holds for some $s > 0$. *Then there are no points conjugate to* t_0 *w.r.t.* $\Lambda(t)$ *in the interval* (t_0, t_0+s).

Proof. Let $\bar{\Lambda}(\tau) = \Lambda(\varphi^{-1}(\tau))$, $\tau_0 = \varphi(t_0)$, and $\tau_s = \varphi(t_0+s)$. By construction, the curve $\bar{\Lambda}(\tau)$ has vanishing Ricci curvature. Denote by $\bar{A}(\tau)$ the density of the fundamental form $\mathcal{A}$, corresponding to the parameter τ. If $A(t) \leq \frac{A_0}{35}$, then by (2.35) $\bar{A}(\tau)$ satisfies the following inequality

$$\bar{A}(\tau) = \frac{A(t)}{\big(\varphi'(t)\big)^4} \leq \frac{A_0}{\min\limits_{t_0 \leq t \leq t_0+s} \big(\varphi'(t)\big)^4} \tag{2.37}$$

on the interval $[\tau_0, \tau_s]$.

Then according to the second statement of Theorem 5, there are no points conjugate to τ_0 w.r.t. the curve $\bar{\Lambda}(\tau)$ in the interval (τ_0, τ_s) if the following inequality holds:

$$\frac{p_1}{\sqrt[4]{A_0}} \min_{t_0 \leq t \leq t_0+s} \varphi'(t) \geq \tau_s - \tau_0 = \varphi(t_0+s) - \varphi(t_0).$$

But this inequality is equivalent to (2.36). This concludes the proof of the lemma. □

Assume that

$$\frac{4}{3} r \leq \rho(t) \leq \frac{4}{3} R. \tag{2.38}$$

Let $y_{q,c}(t)$ be the solution of the following equation with given initial conditions

$$y''(t) + q y(t) = 0, \quad y(0) = 1, \quad y'(0) = c,$$

namely,

$$y_{q,c}(t) = \cos\sqrt{q}\,t + \frac{c}{\sqrt{q}}\sin\sqrt{q}\,t = \\ = \begin{cases} \cosh\sqrt{-q}\,t + \dfrac{c}{\sqrt{-q}}\sinh\sqrt{-q}\,t, & q < 0, \\ \cos\sqrt{q}\,t + \dfrac{c}{\sqrt{q}}\sin\sqrt{q}\,t, & q > 0, \\ 1 + ct, & q = 0. \end{cases} \tag{2.39}$$

Let M_c be the minimal positive zero of the function $y_{R,c}(t)$ if such zero exists and $M_c = \infty$ otherwise. We denote

$$\mathcal{D}_R = \big\{(s,c) : 0 < s < M_c\big\}. \tag{2.40}$$

In the domain $\mathcal{D}_R$, we define the following function:

$$K(s,c) = \Big(\max_{0\le t\le s} y_{r,c}(t)\Big)^2 \int\limits_0^s \frac{dt}{y^2_{R,c}(t)}. \tag{2.41}$$

It is easy to show that the function $s \mapsto K(s,c)$ is a well-defined monotone increasing function on $[0, M_c)$ which obtains on this set all nonnegative values. Thus, for any c there exists a unique solution $s = S_{A_0}(c)$, $0 < S_{A_0}(c) < M_c$, of the equation

$$K(s,c) = \frac{p_1}{\sqrt[4]{A_0}}. \tag{2.42}$$

Then the following lemma holds.

Lemma 2.4. *For any t_0, there are no points conjugate to t_0 w.r.t. $\Lambda(t)$ in the interval $\big(t_0, t_0 + \sup\limits_{c\in\mathbb{R}} S_{A_0}(c)\big)$.*

Proof. Let $\tau = \varphi(t)$ be a reparametrization of $\Lambda(t)$ to the projective parameter τ such that $\varphi'(t_0) = 1$ and $\varphi''(t_0) = -2c$. Then according to Lemma 2.2, the function $y(t) = \dfrac{1}{\sqrt{\varphi'(t)}}$ satisfies the Hill equation with the prescribed initial conditions

$$y'' + \frac{3\rho(t)}{4}y = 0, \quad y(t_0) = 1, \quad y'(t_0) = c.$$

By the classical Sturm comparison theorem about the Hill equation, we have

$$y_{R,c}(t - t_0) \le \frac{1}{\sqrt{\varphi'(t)}} \le y_{r,c}(t - t_0) \tag{2.43}$$

for any $t \in [t_0, t_0 + M_c)$. Therefore, for $0 \le s < M_c$ we obtain

$$\frac{\varphi(t_0+s)-\varphi(t)}{\min\limits_{t_0 \le t \le t_0+s} \varphi'(t)} = \left(\max_{t_0 \le t \le t_0+s} \frac{1}{\varphi'(t)}\right)\Big(\varphi(t_0+s)-\varphi(t)\Big) \le$$

$$\le \left(\max_{0 \le t \le s} y_{r,c}(t)\right)^2 \int_0^s \frac{dt}{y^2_{R,c}(t)} = K(s,c).$$

Consequently, for $s = S_{A_0}(c)$

$$\frac{\varphi(t_0+s)-\varphi(t)}{\min\limits_{t_0 \le t \le t_0+s} \varphi'(t)} \le \frac{p_1}{\sqrt[4]{A_0}}.$$

Hence, according to Lemma 2.3 there are no points conjugate to t_0 w.r.t. $\Lambda(t)$ in the interval $\Big(t_0, t_0+S_{A_0}(c)\Big)$. This concludes the proof of the lemma, since an arbitrary real number can be taken as c. $\square$

Lemma 2.4 shows that in order to find the lower bound for the length of the interval without conjugate points, one can find the maximum of the function $S_{A_0}(c)$ given implicitly by the equation $K(s,c) = \frac{p_1}{\sqrt[4]{A_0}}$. It is not difficult to find the expressions for the function $K(s,c)$. It is convenient to write theem in terms of trigonometric functions. Namely,

(1) if $r > 0$, then

$$K(s,c) = \begin{cases} \dfrac{\Big(\cos(\sqrt{r}\,s) + \frac{c}{\sqrt{r}}\sin(\sqrt{r}\,s)\Big)^2}{\sqrt{R}\cot(\sqrt{R}\,s)+c}, \\ \qquad 0 \le s \le \frac{1}{\sqrt{r}}\arctan\left(\frac{c}{\sqrt{r}}\right); & (2.45a) \\ \dfrac{1+\frac{c^2}{r}}{\sqrt{R}\cot(\sqrt{R}\,s)+c}, \\ s > \frac{1}{\sqrt{r}}\arctan\left(\frac{c}{\sqrt{r}}\right), \quad c \ge 0; & (2.45b) \\ \dfrac{1}{\sqrt{R}\cot(\sqrt{R}\,s)+c}, \\ \qquad c < 0; & (2.45c) \end{cases}$$

(2) if $r \le 0$, then

$$K(s,c) = \begin{cases} \dfrac{\left(\cos(\sqrt{r}\,s) + \frac{c}{\sqrt{r}}\sin(\sqrt{r}\,s)\right)^2}{\sqrt{R}\,\cot(\sqrt{R}\,s) + c}, \\ \qquad c \ge -\sqrt{|r|}\tanh\left(\dfrac{\sqrt{|r|}\,s}{2}\right); & (2.46a) \\ \dfrac{1}{\sqrt{R}\,\cot(\sqrt{R}\,s) + c}, \\ \qquad c < -\sqrt{|r|}\tanh\left(\dfrac{\sqrt{|r|}\,s}{2}\right). & (2.46b) \end{cases}$$

Note that for negative values of r and R the trigonometric functions with imaginary arguments can be replaced in the above formulas by the corresponding hyperbolic functions with real arguments and if $r = 0$ (or $R = 0$), then the above formulas above are replaced by their limits as r (or R) tends to zero.

Consider the cases of positive and nonnegative r separately.

2.4.3. (a) *The case* $r > 0$. On the domain $\mathcal{D}_R$ define the function

$$K_1(s,c) \stackrel{\text{def}}{=} \frac{1 + \dfrac{c^2}{r}}{\sqrt{R}\cot(\sqrt{R}\,s) + c}. \tag{2.47}$$

For any c let $s = S_{1,A_0}(c)$ be the minimal positive solution of the equation

$$K_1(s,c) = \frac{p_1}{\sqrt[4]{A_0}},$$

i.e.,

$$S_{1,A_0}(c) = \frac{1}{\sqrt{R}}\operatorname{arccot}\left(\frac{\sqrt[4]{A_0}}{p_1\sqrt{R}}\left(1 + \frac{c^2}{r} - \frac{p_1}{\sqrt[4]{A_0}}c\right)\right). \tag{2.48}$$

It is clear that in the considered case, the following inequality holds

$$K(c,s) \le K_1(c,s)$$

on the domain $\mathcal{D}_R$. Hence we obtain $S_{1,A_0}(c) \le S_{A_0}(c)$ and

$$\sup_{c\in\mathbb{R}} S_{1,A_0}(c) \le \sup_{c\in\mathbb{R}} S_{A_0}(c). \tag{2.49}$$

From (2.48), it easily follows that

$$\max_{c\in\mathbb{R}} S_{1,A_0}(c) = \frac{1}{\sqrt{R}}\operatorname{arccot}\,\frac{1}{\sqrt{R}}\left(\frac{\sqrt[4]{A_0}}{p_1} - \frac{p_1 r}{4\sqrt[4]{A_0}}\right). \tag{2.50}$$

Applying Lemma 2.4 and (2.49), we obtain the following theorem.

Theorem 7. *Let $\Lambda : I \mapsto L(W)$ be a curve of rank 1 and constant weight defined on the interval $I \subseteq \mathbb{R}$. Suppose that for any $t \in I$ its invariants $\rho(t)$ and $A(t)$ satisfy the inequalities*

$$0 < \frac{4}{3} r \leq \rho(t) \leq \frac{4}{3} R, \quad A(t) \leq \frac{A_0}{35}, \quad A_0 > 0.$$

Then for any t_0 in the interval

$$\left(t_0, t_0 + \frac{1}{\sqrt{R}} \operatorname{arccot}\left(\frac{1}{\sqrt{R}}\left(\frac{\sqrt[4]{A_0}}{p_1} - \frac{p_1 r}{4\sqrt[4]{A_0}}\right)\right)\right) \tag{2.51}$$

there are no points conjugate to t_0 w.r.t. the curve $\Lambda(t)$.

Considering the function

$$K_2(s, c) \stackrel{\text{def}}{=} \frac{\left(\cos(\sqrt{r}\, s) + \frac{c}{\sqrt{r}} \sin(\sqrt{r}\, s)\right)^2}{\sqrt{R} \cot\left(\sqrt{R}\, s\right) + c} \tag{2.52}$$

from (2.45a), one can try to improve the previous theorem. For a given s consider the following equation w.r.t. c:

$$K_2(s, c) = \frac{p_1}{\sqrt[4]{A_0}}. \tag{2.53}$$

This equation is equivalent to the quadratic equation with the discriminant

$$D(s) = \frac{p_1}{\sqrt[4]{A_0}} \left(\frac{p_1}{\sqrt[4]{A_0}} - L_{r,R}(s)\right), \tag{2.54}$$

where

$$L_{r,R}(s) \stackrel{\text{def}}{=} 4\left(\cos(\sqrt{r} s) - \sqrt{\frac{R}{r}} \sin(\sqrt{r}\, s) \cot\left(\sqrt{R}\, s\right)\right) \frac{\sin(\sqrt{r}\, s)}{\sqrt{r}}. \tag{2.55}$$

It is not difficult to show that in the considered case ($R > r > 0$) the function $L_{r,R}(s)$ is monotone increasing on the interval $\left(0, \frac{\pi}{\sqrt{R}}\right)$ and obtains all positive values on this interval and, therefore, the inverse function $L_{r,R}^{-1} : (0, \infty) \mapsto \left(0, \frac{\pi}{\sqrt{R}}\right)$ is well defined. This together with (2.54) implies that the maximal value of s for the points (s,c) on the level set $K_2(s,c) = \frac{p_1}{\sqrt[4]{A_0}}$ is attained at the point $(\tilde{s},\tilde{c})$ such that

$$\tilde{s} = L_{r,R}^{-1}\left(\frac{p_1}{\sqrt[4]{A_0}}\right), \tag{2.56}$$

$$\tilde{c} = \sqrt{r}\cot\Bigl(\sqrt{r}\,\tilde{s}\Bigr) - 2\sqrt{R}\cot\Bigl(\sqrt{R}\,\tilde{s}\Bigr). \tag{2.57}$$

If at the same time

$$\tilde{s} < \frac{\pi}{2\sqrt{r}} \tag{2.58}$$

and

$$\tilde{c} > \sqrt{r}\,\tan(\sqrt{r}\,\tilde{s}), \tag{2.59}$$

then it is easy to see that $\sup_{c\in\mathbb{R}} S_{A_0}(c) = \tilde{s} > \sup_{c\in\mathbb{R}} S_{1,A_0}(c)$ (see (2.45)). In this case we will improve Theorem 7 by setting in (2.51) $\tilde{s}$ instead of $\max_{c\in\mathbb{R}} S_{1,A_0}(c)$. Indeed, if we denote

$$N_{r,R}(s) \stackrel{\text{def}}{=} \sqrt{r}\cot\Bigl(\sqrt{r}\,s\Bigr) - 2\sqrt{R}\cot\Bigl(\sqrt{R}\,s\Bigr) - \sqrt{r}\tan\Bigl(\sqrt{r}\,s\Bigr), \tag{2.60}$$

then we have the following theorem in addition to Theorem 7.

Theorem 8. *Let $\Lambda : I \mapsto L(W)$ be curve of rank 1 and constant weight defined on the interval $I \subseteq \mathbb{R}$. Suppose that for any $t \in I$ its invariants $\rho(t)$ and $A(t)$ satisfy the inequalities*

$$0 < \frac{4}{3}r \le \rho(t) \le \frac{4}{3}R, \quad A(t) \le \frac{A_0}{35}, \quad A_0 > 0.$$

Suppose also that

$$L_{r,R}^{-1}\left(\frac{p_1}{\sqrt[4]{A_0}}\right) < \frac{\pi}{2\sqrt{r}} \tag{2.61}$$

and

$$N_{r,R}\left(L_{r,R}^{-1}\Bigl(\frac{p_1}{\sqrt[4]{A_0}}\Bigr)\right) > 0. \tag{2.62}$$

Then for any t_0 there are no points conjugate to t_0 w.r.t. $\Lambda(t)$ in the interval $\Bigl(t_0, t_0 + L_{r,R}^{-1}\Bigl(\frac{p_1}{\sqrt[4]{A_0}}\Bigr)\Bigr)$.

Note that if $\dfrac{\pi}{2\sqrt{r}} \geq \dfrac{\pi}{\sqrt{R}}$, i.e., $R \geq 4r$, then condition (2.61) holds automatically. Also, one can directly verify that in this case the function $N_{r,R}(s)$ is monotone increasing on the interval $\left(0, \dfrac{\pi}{\sqrt{R}}\right)$ and obtains all real values there. Therefore, for sufficiently small positive A_0 condition (2.62) also holds. Hence for $R \geq 4r$ and sufficiently small positive A_0 Theorem 8 improves Theorem 7.

If $R < 4r$, then the function $N_{r,R}(s)$ tends to $-\infty$ at the points 0 and $\dfrac{\pi}{2\sqrt{r}}$. If $\dfrac{R}{r}$ is sufficiently close to 4, then $N_{r,R}(s)$ is positive on some subinterval of $\left(0, \dfrac{\pi}{2\sqrt{r}}\right)$ and Theorem 8 is relevant for A_0 with $L_{r,R}^{-1}\left(\dfrac{p_1}{\sqrt[4]{A_0}}\right)$ lying in this subinterval. But there exists $\alpha \in (1,4)$ such that for $1 \leq \dfrac{R}{r} < \alpha$ the function $N_{r,R}(s)$ obtains only negative values on $\left(0, \dfrac{\pi}{2\sqrt{r}}\right)$. In this case Theorem 8 is not relevant.

2.4.4. (b) *The case* $r \leq 0$. Suppose that the functions $K_2(s,c)$ and $L_{r,R}(s)$ are as in (2.52) and (2.55) respectively. It is not difficult to show that in the considered case ($r < 0$, $r < R$) the function $L_{r,R}(s)$ is monotone increasing and obtains all positive values on $\left(0, \dfrac{\pi}{\sqrt{R}}\right)$ if $R > 0$, and on $(0, \infty)$, if $R \leq 0$. Therefore, the inverse function $L_{r,R}^{-1}$ is well defined on $(0,\infty)$ (with values in $\left(0, \dfrac{\pi}{\sqrt{R}}\right)$ if $R > 0$, and in $(0,\infty)$ if $R \leq 0$). As in the previous case, the maximal value of s for the points (s,c) on the level set $K_2(s,c) = \dfrac{p_1}{\sqrt[4]{A_0}}$ is attained at the point $(\tilde{s}, \tilde{c})$, satisfying (2.58) and (2.59). Also, we note that for any s the function

$$K_3(s,c) \overset{\text{def}}{=} \frac{1}{\sqrt{R}\,\cot\left(\sqrt{R}\,s\right) + c},$$

which appears in (2.45b), is a monotone decreasing function of c. Therefore, if $\tilde{c} > -\sqrt{r}\tan\left(\dfrac{\sqrt{r}\,\tilde{s}}{2}\right)$, then $\sup\limits_{c\in\mathbb{R}} S_{A_0}(c) = \tilde{s}$ and if $\tilde{c} \leq -\sqrt{r}\tan\left(\dfrac{\sqrt{r}\,\tilde{s}}{2}\right)$, then $s = \sup\limits_{c\in\mathbb{R}} S_{A_0}(c)$ satisfies the following equation:

$$K_3\left(s, -\sqrt{|r|}\tanh\left(\frac{\sqrt{|r|}\,s}{2}\right)\right) = \frac{p_1}{\sqrt[4]{A_0}} \tag{2.63}$$

(see relations (2.45)).

Denote

$$Q_{r,R}(s) \overset{\text{def}}{=} K_3\left(s, -\sqrt{|r|}\tanh\left(\frac{\sqrt{|r|}\,s}{2}\right)\right) = \\ = \frac{1}{\sqrt{R}\cot\left(\sqrt{R}\,s\right) - \sqrt{|r|}\tanh\left(\frac{\sqrt{|r|}\,s}{2}\right)} \tag{2.64}$$

and

$$N_{r,R}(s) \overset{\text{def}}{=} \sqrt{r}\cot\left(\sqrt{r}\,s\right) - 2\sqrt{R}\cot\left(\sqrt{R}\,s\right) + \\ + \sqrt{|r|}\tanh\left(\frac{\sqrt{|r|}\,s}{2}\right), \quad r \leq 0. \tag{2.65}$$

Obviously, the function $Q_{r,R}(s)$ is monotone increasing and obtains all positive values on some interval of the form $(0, S_3)$. We obtain the following theorem.

Theorem 9. *Let $\Lambda : I \mapsto L(W)$ be curve of rank 1 and constant weight defined on the interval $I \subseteq \mathbb{R}$. Suppose that for any $t \in I$ its invariants $\rho(t)$ and $A(t)$ satisfy the inequalities*

$$\frac{4}{3}r \leq \rho(t) \leq \frac{4}{3}R, \quad r \leq 0, \quad A(t) \leq \frac{A_0}{35}, \quad A_0 > 0.$$

Then the following two statements hold:

(1) *if*

$$N_{r,R}\left(L_{r,R}^{-1}\left(\frac{p_1}{\sqrt[4]{A_0}}\right)\right) > 0, \tag{2.66}$$

then for any t_0 there are no points conjugate to t_0 w.r.t. $\Lambda(t)$ in the interval

$$\left(t_0, t_0 + L_{r,R}^{-1}\left(\frac{p_1}{\sqrt[4]{A_0}}\right)\right); \tag{2.67}$$

(2) *if*

$$N_{r,R}\left(L_{r,R}^{-1}\left(\frac{p_1}{\sqrt[4]{A_0}}\right)\right) \leq 0 \tag{2.68}$$

then for any t_0 there are no points conjugate to t_0 w.r.t. $\Lambda(t)$ in the interval

$$\left(t_0, t_0 + Q_{r,R}^{-1}\left(\frac{p_1}{\sqrt[4]{A_0}}\right)\right). \tag{2.69}$$

It is not difficult to show that for $r \leq 0$ the function $N_{r,R}(s)$ is monotone increasing on $\left(0, \frac{\pi}{\sqrt{R}}\right)$ if $R > 0$, and on $(0, \infty)$ if $R \leq 0$. Moreover, for $R > 0$ the function $N_{r,R}(s)$ obtains all real values on $\left(0, \frac{\pi}{\sqrt{R}}\right)$ and for $R \leq 0$ the function $N_{r,R}(s)$ obtains on $(0, \infty)$ all real values less than $2(\sqrt{|r|} - \sqrt{|R|})$, which is a positive number. Therefore, in both cases sufficiently small positive A_0 correspond to the first item of the previous theorem, whenever sufficiently large A_0 correspond to the second item.

2.4.5. *The case* $A(t) \geq \frac{A_0}{35} > 0$. First, by analogy with Lemma 2.3, we prove the following auxiliary lemma.

Lemma 2.5. *Let $\tau = \varphi(t)$ be a reparametrization of the curve $\Lambda(t)$ to the projective parameter τ such that the function $\varphi(t)$ is monotone increasing. Assume that $A(t) \geq \frac{A_0}{35}$ and that the inequality*

$$\frac{\varphi(t_0 + s) - \varphi(t)}{\max\limits_{t_0 \leq t \leq t_0 + s} \varphi'(t)} \geq \frac{p_1}{\sqrt[4]{A_0}} \tag{2.70}$$

holds for some $s > 0$. Then there exists at least one point conjugate to t_0 w.r.t. $\Lambda(t)$ in the interval $(t_0, t_0 + s]$.

Proof. Let $\bar{\Lambda}(\tau) = \Lambda(\varphi^{-1}(\tau))$, $\tau_0 = \varphi(t_0)$, $\tau_s = \varphi(t_0 + s)$. By construction, the curve $\bar{\Lambda}(\tau)$ has vanishing Ricci curvature. Denote by $\bar{A}(\tau)$ the density of the fundamental form $\mathcal{A}$, corresponding to the parameter τ. If $A(t) \geq \frac{A_0}{35}$, then by (2.35) $\bar{A}(\tau)$ satisfies the inequality

$$\bar{A}(\tau) = \frac{A(t)}{\big(\varphi'(t)\big)^4} \geq \frac{A_0}{\max\limits_{t_0 \leq t \leq t_0 + s} \big(\varphi'(t)\big)^4} \tag{2.71}$$

on the interval $[\tau_0, \tau_s]$. Then according to item 3 of Theorem 5, there exists at least one point conjugate to τ_0 w.r.t. the curve $\bar{\Lambda}(\tau)$ in the interval $(\tau_0, \tau_s]$ if the following inequality holds:

$$\frac{p_1}{\sqrt[4]{A_0}} \max_{t_0 \leq t \leq t_0 + s} \varphi'(t) \leq \tau_s - \tau_0 = \varphi(t_0 + s) - \varphi(t_0).$$

But this inequality is equivalent to (2.70). This concludes the proof of the lemma. □

Assume again that

$$\frac{4}{3} r \leq \rho(t) \leq \frac{4}{3} R \tag{2.72}$$

and that $y_{q,c}(t)$ is as in (2.39). As before, let M_c be the minimal positive zero of the function $y_{R,c}(t)$ if such a zero exists and $M_c = \infty$ otherwise. Also, let $\mathcal{D}_R$ be as in (2.40). On the domain $\mathcal{D}_R$ we define the function

$$\overline{K}(s,c) = \left(\min_{0\le t\le s} y_{R,c}(t)\right)^2 \int_0^s \frac{dt}{y_{r,c}^2(t)}. \tag{2.73}$$

By analogy with Lemma 2.4, we have the following lemma.

Lemma 2.6. *If for some $s > 0$ there exists c such that $(s,c) \in \mathcal{D}_R$ and*

$$\overline{K}(s,c) \ge \frac{p_1}{\sqrt[4]{A_0}}, \tag{2.74}$$

then for any t_0 there exists at least one point conjugate to t_0 w.r.t. $\Lambda(t)$ in the interval $(t_0, t_0 + s]$.

Proof. Let $\tau = \varphi(t)$ be a reparametrization of $\Lambda(t)$ to the projective parameter τ such that $\varphi(t_0) = 1$ and $\varphi'(t_0) = -2c$. Then, using (2.43), we have

$$\frac{\varphi(t_0+s)-\varphi(t)}{\max\limits_{t_0\le t\le t_0+s}\varphi'(t)} = \left(\min_{t_0\le t\le t_0+s}\frac{1}{\varphi'(t)}\right)\Big(\varphi(t_0+s)-\varphi(t)\Big) \ge$$

$$\ge \left(\min_{0\le t\le s} y_{R,c}(t)\right)^2 \int_0^s \frac{dt}{y_{r,c}^2(t)} = \overline{K}(s,c).$$

Consequently, if $\overline{K}(s,c) \ge \dfrac{p_1}{\sqrt[4]{A_0}}$, then

$$\frac{\varphi(t_0+s)-\varphi(t)}{\max\limits_{t_0\le t\le t_0+s}\varphi'(t)} \ge \frac{p_1}{\sqrt[4]{A_0}}.$$

Hence, according to Lemma 2.5 there exists at least one point conjugate to t_0 w.r.t. $\Lambda(t)$ in the interval (t_0, t_0+s). This concludes the proof of the lemma. □

Lemma 2.6 shows that in order to find the upper bound for the next conjugate point, one can find the minimum of the function $s = \overline{S}_{A_0}(c)$ given implicitly by the equation $\overline{K}(s,c) = \dfrac{p_1}{\sqrt[4]{A_0}}$, where $0 < s < M_c$. In contrast to the case considered in Subsec. 8.4.2 (where we analyze the level sets of the function $K(s,c)$), the level sets $\overline{K}(s,c) = \dfrac{p_1}{\sqrt[4]{A_0}}$ can be empty for sufficiently small positive A_0 and in this case we are not able to estimate the next conjugate point. Also, in general, the function $\overline{S}_{A_0}(c)$ for some c is not defined or can be multivalued.

It is not difficult to find the expressions for the function $\overline{K}(s,c)$. Namely,

(1) if $R<0$, then

$$\overline{K}(s,c)=\begin{cases}\dfrac{\left(\cosh(\sqrt{|R|}\,s)+\frac{c}{\sqrt{|R|}}\sinh(\sqrt{|R|}\,s)\right)^2}{\sqrt{|r|}\,\coth(\sqrt{|r|}\,s)+c}, & \\ \qquad c\le -\sqrt{|R|}\tanh(\sqrt{|R|}\,s); & (2.75a)\\ \dfrac{1+\frac{c^2}{R}}{\sqrt{|r|}\coth(\sqrt{|r|}\,s)+c}, & \\ \qquad -\sqrt{|R|}\tanh\sqrt{|R|}\,s\le c\le 0; & (2.75b)\\ \dfrac{1}{\sqrt{|r|}\coth(\sqrt{|r|}\,s)+c}, & \\ \qquad c>0; & (2.75c)\end{cases}$$

(2) if $R\ge 0$, then

$$\overline{K}(s,c)=\begin{cases}\dfrac{\left(\cos(\sqrt{R}\,s)+\frac{c}{\sqrt{R}}\sin(\sqrt{R}\,s)\right)^2}{\sqrt{r}\cot(\sqrt{r}\,s)+c}, & \\ \qquad c\le\sqrt{R}\tan\left(\dfrac{\sqrt{R}\,s}{2}\right); & (2.76a)\\ \dfrac{1}{\sqrt{r}\cot(\sqrt{r}\,s)+c}, & \\ \qquad c>\sqrt{R}\tan\left(\dfrac{\sqrt{R}\,s}{2}\right); & (2.76b)\end{cases}$$

Consider the cases of negative and nonnegative R separately.

2.4.6. (a) *The case of $R<0$.*

Theorem 10. *Let $\Lambda: I\mapsto L(W)$ be a curve of rank 1 and constant weight defined on the interval $I\subseteq\mathbb{R}$. Suppose that for any $t\in I$ its invariants $\rho(t)$ and $A(t)$ satisfy the inequalities*

$$\frac{4}{3}r\le\rho(t)\le\frac{4}{3}R<0,\quad A(t)\ge\frac{A_0}{35}>0.$$

Suppose also that the bounds A_0, r, and R satisfy the relation

$$\frac{2}{\sqrt{|r|}+\sqrt{|r|-|R|}}>\frac{p_1}{\sqrt[4]{A_0}}. \tag{2.77}$$

Then for any t_0 there exists at least one point conjugate to t_0 w.r.t. the curve $\Lambda(t)$ in the interval

$$\left(t_0, t_0 + \frac{1}{\sqrt{|r|}}\operatorname{arccoth}\left(\frac{1}{\sqrt{|r|}}\left(\frac{\sqrt[4]{A_0}}{p_1} - \frac{p_1 R}{4\sqrt[4]{A_0}}\right)\right)\right]. \tag{2.78}$$

Proof. On the domain $\mathcal{D}_R$, we define the following function:

$$\overline{K_1}(s,c) \stackrel{\text{def}}{=} \frac{1+\dfrac{c^2}{R}}{\sqrt{r}\cot\left(\sqrt{r}\,s\right)+c} = \frac{1+\dfrac{c^2}{R}}{\sqrt{|r|}\coth\left(\sqrt{|r|}\,s\right)+c}. \tag{2.79}$$

From the elementary properties of the hyperbolic functions one can obtain the following inequality:

$$\overline{K}(s,c) \geq \overline{K_1}(s,c). \tag{2.80}$$

Therefore, if the point $(s,c) \in \mathcal{D}_R$ and belongs to the level set

$$\overline{K_1}(s,c) = \frac{p_1}{\sqrt[4]{A_0}}, \tag{2.81}$$

then by Lemma 2.6, for any t_0 there exists at least one point conjugate to t_0 w.r.t. $\Lambda(t)$ in the interval $(t_0, t_0+s]$. First, by a direct computation, one can easily show that

$$\sup_{\mathcal{D}_R} \overline{K_1}(s,c) = \frac{2}{\sqrt{|r|}+\sqrt{|r|-|R|}} \tag{2.82}$$

(but at the same time $\overline{K_1}(s,c) < \dfrac{2}{\sqrt{|r|}+\sqrt{|r|-|R|}}$ for all $(s,c) \in \mathcal{D}_R$). Hence the level set (2.81) is not empty iff the bounds A_0, r, and R satisfy inequality (2.77).

Now suppose that (2.77) holds and for a given c consider the equation (2.81) as the equation w.r.t. s. Obviously, this equation is equivalent to the following one:

$$\coth(\sqrt{|r|}s) = \frac{\sqrt[4]{A_0}}{p_1\sqrt{|r|}}\left(1+\frac{c^2}{R} - \frac{p_1}{\sqrt[4]{A_0}}c\right). \tag{2.83}$$

Denote the right-hand side of the last equation by $\sigma(c)$. Since for $x>0$ the function $\coth(x)$ is monotone decreasing, Eq. (2.83) has a minimal positive solution $s_{\min}$ for $c = \dfrac{p_1 R}{2\sqrt[4]{A_0}}$, corresponding to the global maximum of the function $\sigma(c)$. It is easy to see also that

$$\max \sigma(c) = \frac{1}{\sqrt{|r|}}\left(\frac{\sqrt[4]{A_0}}{p_1} - \frac{p_1 R}{4\sqrt[4]{A_0}}\right).$$

Inequality (2.77) holds then

$$s_{\min} = \frac{1}{\sqrt{|r|}} \operatorname{arccoth}\left(\frac{1}{\sqrt{|r|}}\left(\frac{\sqrt[4]{A_0}}{p_1} - \frac{p_1 R}{4\sqrt[4]{A_0}}\right)\right). \tag{2.84}$$

Further from (2.80) it follows that if condition (2.77) holds, then we have

$$\overline{K}\left(s_{\min}, \frac{p_1 R}{2\sqrt[4]{A_0}}\right) \geq \overline{K}_1\left(s_{\min}, \frac{p_1 R}{2\sqrt[4]{A_0}}\right) = \frac{p_1}{\sqrt[4]{A_0}}.$$

Therefore, by Lemma 2.6 there exists at least one point conjugate to t_0 w.r.t. $\Lambda(t)$ in the interval $(t_0, t_0 + s_{\min}]$. This together with (2.84) completes the proof of the theorem. □

Remark 8. It is not difficult to show that the relation

$$\sup_{\mathcal{D}_r} \overline{K}(s,c) = \sup_{\mathcal{D}_R} \overline{K_1}(s,c)$$

holds. In addition, the minimal value of s for the points (s,c) on the level set $\overline{K}(s,c) = \frac{p_1}{\sqrt[4]{A_0}}$ is equal to the minimal value of s for the points (s,c) on the level set $\overline{K_1}(s,c) = \frac{p_1}{\sqrt[4]{A_0}}$. This shows that in spite of the inequality (2.80), the replacement of the function $\overline{K}(s,c)$ by the function $\overline{K_1}(s,c)$ in the proof of the previous theorem leads to the same estimate for the length of intervals containing conjugate points.

2.4.7. (b) *The case* $R \geq 0$. On the interval $\left(0, \frac{\pi}{\sqrt{R}}\right)$, we define the function

$$\overline{Q}_{r,R}(s) \stackrel{\text{def}}{=} \overline{K}\left(s, \sqrt{R}\tan\left(\frac{\sqrt{R}\,s}{2}\right)\right) = \\ = \frac{1}{\sqrt{r}\cot\left(\sqrt{r}\,s\right) + \sqrt{R}\tan\left(\frac{\sqrt{R}\,s}{2}\right)}. \tag{2.85}$$

Note that $\overline{Q}_{r,R}(s) > 0$ on the interval $\left(0, \frac{\pi}{\sqrt{R}}\right)$. Also, $\overline{Q}_{r,R}(s) \underset{s\to 0}{\longrightarrow} 0$ and $\overline{Q}_{r,R}(s) \underset{s\to\frac{\pi}{\sqrt{R}}}{\longrightarrow} 0$. Therefore, there exists $s_{\max} \in \left(0, \frac{\pi}{\sqrt{R}}\right)$ such that $\overline{Q}_{r,R}(s)$ attains its maximum at $s = s_{\max}$. Moreover, it is not difficult to show that on the interval $(0, s_{\max}]$ the function $\overline{Q}_{r,R}(s)$ is monotone increasing. Therefore, the inverse function $\overline{Q}_{r,R}^{-1} : (0, \overline{Q}_{r,R}(s_{\max})] \mapsto (0, s_{\max}]$ (or the branch of the multi-valued function inverse to $\overline{Q}_{r,R}$ that obtains its values on $(0, s_{\max}]$) is well defined.

Theorem 11. *Let $\Lambda : I \mapsto L(W)$ be a curve of rank 1 and constant weight defined on the interval $I \subseteq \mathbb{R}$. Suppose that for any $t \in I$ its invariants $\rho(t)$ and $A(t)$ satisfy the inequalities*

$$\frac{4}{3}r \le \rho(t) \le \frac{4}{3}R, \quad R \ge 0, \quad A(t) \ge \frac{A_0}{35} > 0.$$

Suppose also that the bounds A_0, r, and R satisfy the relation

$$\max_{0<s<\frac{\pi}{\sqrt{R}}} \overline{Q}_{r,R}(s) \ge \frac{p_1}{\sqrt[4]{A_0}}. \tag{2.86}$$

Then for any t_0 there exists at least one point conjugate to t_0 w.r.t. the curve $\Lambda(t)$ in the interval

$$\left(t_0, t_0 + \overline{Q}_{r,R}^{-1}\left(\frac{p_1}{\sqrt[4]{A_0}}\right)\right]. \tag{2.87}$$

Proof. Note that in our case the domain $\mathcal{D}_R$ has the form

$$\mathcal{D}_R = \left\{(s,c) : c > -\sqrt{R}\cot(\sqrt{R}\,s)\right\}.$$

For given s, $0 < s < \dfrac{\pi}{\sqrt{R}}$, consider the function $c \mapsto \overline{K}(s,c)$, where $c > -\sqrt{R}\cot(\sqrt{R}\,s)$. From relations (2.76) it follows that $\overline{K}(s,c) \longrightarrow 0$ as $c \to -\sqrt{R}\cot(\sqrt{R}\,s)$ or $c \to \infty$. Also, we note that for $0 < s < \dfrac{\pi}{\sqrt{R}}$ the inequality

$$\sqrt{R}\,\tan\left(\frac{\sqrt{R}\,s}{2}\right) > -\sqrt{R}\,\cot(\sqrt{R}\,s)$$

holds. Obviously, the function $c \mapsto \overline{K}(s,c)$ is monotone decreasing for $c > \sqrt{R}\,\tan\left(\dfrac{\sqrt{R}\,s}{2}\right)$. Also, by a direct calculation we can easily obtain that the function $c \mapsto \overline{K}(s,c)$ is monotone increasing for $-\sqrt{R}\cot(\sqrt{R}\,s) < c < \sqrt{R}\,\tan\left(\dfrac{\sqrt{R}\,s}{2}\right)$. Hence for all $(s,c) \in \mathcal{D}_R$ we have the following relation:

$$\overline{K}(s,c) \le \overline{K}\left(s, \sqrt{R}\,\tan\left(\frac{\sqrt{R}\,s}{2}\right)\right) = Q_{r,R}(s). \tag{2.88}$$

Consequently,

$$\sup_{\mathcal{D}_R} \overline{K}(s,c) = \max_{0<s<\frac{\pi}{\sqrt{R}}} Q_{r,R}(s),$$

which implies that the level set $\overline{K}(s,c) = \dfrac{p_1}{\sqrt[4]{A_0}}$ is not empty iff the bounds A_0, r, and R satisfy inequality (2.86).

Now suppose that (2.86) holds. If $s < \overline{Q}_{r,R}^{-1}\Big(\frac{p_1}{\sqrt[4]{A_0}}\Big)$, then from monotonicity of $Q_{r,R}$ on the interval $(0, s_{\max})$ and relation (2.88) it follows that

$$\overline{K}(s,c) < Q_{r,R}(s) < \frac{p_1}{\sqrt[4]{A_0}}.$$

On the other hand, if $s = \overline{Q}_{r,R}^{-1}\Big(\frac{p_1}{\sqrt[4]{A_0}}\Big)$, then $K\Big(s, \sqrt{R}\tan\Big(\frac{\sqrt{R}\,s}{2}\Big)\Big) = \frac{p_1}{\sqrt[4]{A_0}}$. Therefore, $s = \overline{Q}_{r,R}^{-1}\Big(\frac{p_1}{\sqrt[4]{A_0}}\Big)$ is the minimal value of s for the points (s,c) on the level set $\overline{K}(s,c) = \frac{p_1}{\sqrt[4]{A_0}}$. Our theorem follows now from Lemma 2.6. $\square$

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(Received 24.02.2002)

Authors' addresses:
A. Agrachev
S.I.S.S.A., Via Beirut 2-4,
34013 Trieste, Italy
and
Steklov Mathematical Institute,
Moscow, Russia
E-mail: agrachev@sissa.it

I. Zelenko
Department of Mathematics, Technion,
Haifa, 32000, Israel
E-mail: zigor@techunix.technion.ac.il