2023 Power Team
Texas A&M High School Students Contest
November 2023

- Each power team entry must have a cover sheet (**typed**). The cover sheet must contain the full name of the school (don't abbreviate), the coach's name, contact email, physical address (so that we know where to send the prizes in case your team wins), the team's name, and the names of each team member. For example, if a school has 3 power team entries, then there should be three different team names. The following is an example of an acceptable title page:

  2023 Power Team Entry
  XXXX School, Team 1
  Coach: Ms. Wizard
  Contact email:
  Address:
  Team Members:
  Jane Doe
  John Smith
  etc.

- Team participants are not allowed to consult with anyone but their teammates. Participants are not allowed to look on the web (including artificial intelligence tools) for any information regarding the power team exam, nor are they allowed to search books or other reference materials.
- Team entries are expected to be neat and legible. If not, they face the possibility of being disqualified by the judges.
- It is expected that you show all work clearly presenting all steps and/or calculations done that lead to the final answer/conclusion.
- **Submissions:**
  - **Hand delivered** submissions are due by 9:00 am on the day of the contest. We will have two boxes for collecting these exams (one on the registration table and another one in the coaches breakroom).
  - **Online submission** should be done no later than 8:30 AM on the day of the contest (Saturday, November 4). Send your solution to our new email **math-hsmc@tamu.edu** (you can equally use the previous email **hsmc@math.tamu.edu**) and make your school's and team's names as your email's subject line. Make sure the solution file is

in PDF format (if your solution is handwritten, scan it first and save it as a **single** PDF file).

In case you need assistance with the exam submission, contact Igor Zelenko (email: **zelenko@math.tamu.edu**).

GOOD LUCK!

1. The city map is an infinite square grid of streets: horizontal lines are streets that go in the east-west direction, and vertical lines are streets that go in the north-south direction. A car A starts at the point $(0,0)$ and turns north or east on each crossing with probability $1/2$. A car B starts at the point (n,m), $n,m>0$, and turns south or west on each crossing with probability $1/2$.

   The speeds of the cars are equal and they start simultaneously. Find the probability of the event that they meet, i.e. appear at the same crossing simultaneously.

   You get bonus points (up to a half of the value of the problem) if you find the simplest possible expression for the answer in terms of the binomial coefficients.

2. The town map is a grid of streets $3m \times m$ (there are $m+1$ streets that go in the east-west direction, and $3m+1$ streets that go in the north-south direction). Two cars A and B start at the points $(0,0)$ and $(3m,m)$, respectively, and move in the same way as in the previous problem, with one additional rule: if the car A reaches the northern edge of the grid, it turns to the east and continues to the east (without additional turns); if the car B reaches the southern edge, it turns to the west and continues to the west ((without additional turns).

   Find the probability of the event that they meet.

   You get bonus points (up to a half of the value of the problem) if you find the simplest possible expression for the answer in terms of the binomial coefficients.

In all problems below the taxicab distance between points A,B with coordinates (x_1,y_1) and (x_2,y_2) is $d(A,B)=|x_1-x_2|+|y_1-y_2|$.

3. Suppose that the points $A=(0,0), B=(b_1,b_2)$ satisfy $b_1,b_2\geq 0$. Describe the locus of points C such that $d(A,C)=d(C,B)$ (the "taxicab perpendicular bisector" to AB).

4. Given two points A,B above the line $y=kx, 0<k<1$, find all points C on the line $y=kx$ such that the distance $\mathrm{dist}(A,C)+\mathrm{dist}(B,C)$ is as small as possible.

5. Suppose that a triangle ABC is taxicab equilateral: $d(A,B) = d(B,C) = d(C,A)$. Show that one of the sides of ABC is vertical, horizontal, or has a slope ± 1.

6. For a taxicab equilateral triangle ABC,

   (a) prove that there always exists a point O such that $d(O,A) = d(O,B) = d(O,C)$, i.e., we can always inscribe an equilateral triangle in a taxicab circle;

   (b) Provide an example of a taxicab equilateral triangle such that a point O with this property is not unique.

7. Describe all triangles ABC such that we cannot inscribe ABC in a taxicab circle, i.e., there is no point O with the property $d(O,A) = d(O,B) = d(O,C)$.

8. Points $A_1, \ldots, A_n$, $n \geq 4$, satisfy the following: $1 = d(A_1, A_2) = d(A_2, A_3) = \cdots = d(A_n, A_1)$, and there exists a point O such that $d(O, A_1) = d(O, A_2) = \cdots = d(O, A_n) = r$. Some of the points A_k may coincide. For every n, find the minimal possible value of r.

9. For distinct points $A_1, \ldots, A_n$, $n \geq 4$, assume that the polygon $A_1 \ldots A_n$ is non-self-intersecting and satisfies the same requirements as in problem 8 . For every n, find the minimal possible value of r.

10. For distinct points $A_1, \ldots, A_n$, $n \geq 4$, assume that the polygon $A_1 \ldots A_n$ is non-self-intersecting and satisfies the same requirements as in problem 8 . For every n, find the maximal possible value of r.