Fundamental form and Cartan's tensor of (2,5)-distributions coincide

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Abstract

In our previous paper [12] for generic rank 2 vector distributions on n-dimensional manifold ($n \geq 5$) we constructed a special differential invariant, the *fundamental form*. In the case $n = 5$ this differential invariant has the same algebraic nature, as the *covariant binary biquadratic form*, constructed by E.Cartan in [7], using his "reduction- prolongation" procedure (we call this form Cartan's tensor). In the present paper we prove that our fundamental form coincides (up to constant factor -35) with Cartan's tensor. This result explains geometric reason for existence of Cartan's tensor (originally this tensor was obtained by very sophisticated algebraic manipulations) and gives the true analogs of this tensor in Riemannian geometry. In addition, as a part of the proof, we obtain a new useful formula for Cartan's tensor in terms of structural functions of any frame naturally adapted to the distribution.

Key words: nonholonomic distributions, Pfaffian systems, differential invariants, abnormal extremals, Jacobi curves, Lagrange Grassmannian.

1 Introduction

Rank l vector distribution D on the n-dimensional manifold M or (l, n)-distribution (where $l < n$) is by definition a l-dimensional subbundle of the tangent bundle TM. In other words, for each point $q \in M$ a l-dimensional subspace $D(q)$ of the tangent space T_qM is chosen and $D(q)$ depends smoothly on q. Two vector distributions D_1 and D_2 are called equivalent, if there exists a diffeomorphism $F : M \mapsto M$ such that $F_*D_1(q) = D_2(F(q))$ for any $q \in M$. Two germs of vector distributions D_1 and D_2 at the point $q_0 \in M$ are called equivalent, if there exist neighborhoods U and $\tilde{U}$ of q_0 and a diffeomorphism $F : U \mapsto \tilde{U}$ such that

$$F_*D_1(q) = D_2(F(q)), \ \forall q \in U;$$
$$F(q_0) = q_0.$$

An obvious (but very rough in the most cases) invariant of distribution D at q is so-called *small growth vector* at q: it is the tuple

$$\big(\dim D(q), \dim D^2(q), \dim D^3(q), \dots\big),$$

where D^j is the j-th power of the distribution D, i.e., $D^j = D^{j-1} + [D, D^{j-1}]$. A simple counting of the "number of parameters" in the considered equivalence problem shows that for $l = 2$ the functional invariant should appear starting from $n = 5$. It is well known that in the

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low dimensions $n = 3$ or 4 all generic germs of rank 2 distributions are equivalent. (Darboux's theorem in the case $n = 3$, small growth vector $(2, 3)$ and Engel's theorem in the case $n = 4$, small growth vector $(2, 3, 4)$, see, for example, [5], [13]).

In our previous paper [12], using the notion of Jacobi curve of singular extremals, introduced in [1], and general theory of unparametrized curves in Lagrange Grassmannian, developed in [3] and [4], we constructed a special differential invariant, the *fundamental form* of generic rank 2 vector distributions D on n-dimensional manifold ($n \geq 5$). In the case $n = 5$ this invariant can be realized as invariant homogeneous polynomial of degree 4 on each plane $D(q)$ (in [12] we called this realization the *tangential fundamental form*). Our tangential fundamental form has the same algebraic nature as the *covariant binary biquadratic form*, constructed by E.Cartan in [7], using his "reduction- prolongation" procedure (we call this form *Cartan's tensor*). In the present paper we prove that our fundamental form coincides (up to constant factor -35) with Cartan's tensor. Since these two invariants were constructed in completely different ways, the comparison of them turned to be not so easy task.

The paper is organized as follows. In section 2, following [12], we describe the construction of the fundamental form of (2,5)-distribution with the small growth vector $(2, 3, 5)$. In section 3, following chapter VI of the original paper [7], we briefly describe the main steps of construction of Cartan's tensor, rewriting all formulas that we need for the comparison of our and Cartan's invariants and formulate the main theorem of our paper (Theorem 2). In section 4 for given frame, naturally adapted to the distribution, we derive the formula for fundamental form in terms of its structural functions. This formula is important not only in the proof of our main theorem: in many cases it is more efficient from computational point of view than the method given in [12] (see Theorem 2 there). Finally in section 5 we prove our main theorem. For this we just express our fundamental form in terms of structural functions of special adapted frame, distinguished by Cartan during his reduction process.

In order to find the analog of Cartan's tensor in Riemannian geometry, one can apply the scheme

$$\text{extremals} \ \rightarrow\ \text{Jacobi curves} \ \rightarrow\ \text{fundamental form of Jacobi curves}$$

to the (pseudo)-Riemannian metric and try to compare the obtained invariant with some classical invariants of Riemannian geometry. It was natural to expect that the obtained in this way invariant should be somehow related with Weyl conformal tensor of the metric. The exact relation was obtained by A. Agrachev and it will be presented in one of his forthcoming papers (compare with conjecture in [6], last paragraph of subsection 3.3 there).

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2 Fundamental form of (2,5)-distribution

In this section we describe the construction of the fundamental form of $(2, 5)$-distribution with the small growth vector $(2, 3, 5)$. The presentation is closed to [12].

2.1 Jacobi curve of abnormal extremals. First for the distribution D of the considered class one can distinguish special (unparametrized) curves in the cotangent bundle T^*M of M. For this let $\pi : T^*M \mapsto M$ be the canonical projection. Let σ be standard symplectic structure on T^*M, namely, for any $\lambda \in T^*M$, $\lambda = (p, q)$, $q \in M$,$p \in T_q^*M$ let

$$\sigma(\lambda)(\cdot) = -d\,p\big(\pi_*(\cdot)\big) \tag{2.1}$$

(here we prefer the sign "-" in the right handside , although usually one defines the standard symplectic form on T^*M without this sign). Denote by $(D^l)^\perp \subset T^*M$ the annihilator of the lth power D^l, namely

$$(D^l)^\perp = \{(q,p) \in T^*M : \; p \cdot v = 0 \; \forall v \in D^l(q)\}. \tag{2.2}$$

The set $D^\perp$ is codimension 2 submanifold of T^*M. Consider the restriction $\sigma|_{D^\perp}$ of the form σ on $D^\perp$. It is not difficult to check that (see, for example [11], section 2): the set of points, where the form $\sigma|_{D^\perp}$ is degenerated, coincides with $(D^2)^\perp$; the set $(D^2)^\perp \backslash (D^3)^\perp$ is codimension 1 submanifold of $D^\perp$; for each $\lambda \in (D^2)^\perp \backslash (D^3)^\perp$ the kernel of $\sigma|_{D^\perp}(\lambda)$ is two-dimensional subspace of $T_\lambda D^\perp$, which is transversal to $T_\lambda (D^2)^\perp$. Hence $\forall \lambda \in (D^2)^\perp \backslash (D^3)^\perp$ we have

$$\ker \sigma|_{(D^2)^\perp}(\lambda) = \ker \sigma|_{D^\perp}(\lambda) \cap T_\lambda (D^2)^\perp$$

It implies that these kernels form line distribution in $(D^2)^\perp \backslash (D^3)^\perp$ and define a *characteristic 1-foliation* Ab_D of $(D^2)^\perp \backslash (D^3)^\perp$. Leaves of this foliation will be called *characteristic curves* of distribution D. Actually these characteristic curves are so-called *abnormal extremals* of D. Projections of the characteristic curves to the base manifold M will be called *abnormal trajectories* of D. Conversely, an abnormal extremal projected to the given abnormal trajectory will be called its *lift*.

Remark 2.1 Note that in the considered case $(D^3)^\perp$ coincides with the zero section of T^*M.$\Box$

Further, for a given segment γ of characteristic curve one can construct a special (unparametrized) curve of Lagrangian subspaces, called Jacobi curve, in the appropriate symplectic space. For this for any $\lambda \in (D^2)^\perp$ denote by $\mathcal{J}(\lambda)$ the following subspace of $T_\lambda (D^2)^\perp$

$$\mathcal{J}(\lambda) = \big(T_\lambda (T^*_{\pi(\lambda)} M) + \ker \sigma|_{D^\perp}(\lambda)\big) \cap T_\lambda (D^2)^\perp. \tag{2.3}$$

Here $T_\lambda (T^*_{\pi(\lambda)} M)$ is tangent to the fiber $T^*_{\pi(\lambda)} M$ at the point λ (or vertical subspace of $T_\lambda (T^*M)$). Actually in the considered case $\mathcal{J}$ is rank 4 distribution on the manifold $(D^2)^\perp \backslash (D^3)^\perp$.

Let O_γ be a neighborhood of γ in $(D^2)^\perp$ such that

$$N = O_\gamma / (Ab_D|_{O_\gamma}) \tag{2.4}$$

is a well-defined smooth manifold. The quotient manifold N is a symplectic manifold endowed with a symplectic structure $\bar\sigma$ induced by $\sigma|_{(D^2)^\perp}$. Let $\phi : O_\gamma \to N$ be the canonical projection on the factor. It is easy to check that $\phi_*\big(\mathcal{J}(\lambda)\big)$ is a Lagrangian subspace of the symplectic space $T_\gamma N$, $\forall \lambda \in \gamma$. Let $L(T_\gamma N)$ be the Lagrangian Grassmannian of the symplectic space $T_\gamma N$, i.e.,

$$L(T_\gamma N) = \{\Lambda \subset T_\gamma N : \Lambda^\angle = \Lambda\},$$

where $\Lambda^\angle$ is the skew-symmetric complement of the subspace Λ,

$$\Lambda^\angle = \{v \in T_\gamma N : \bar\sigma(v, \Lambda) = 0\}.$$

Jacobi curve of the characteristic curve (abnormal extremal) γ is the mapping

$$\lambda \mapsto J_\gamma(\lambda) \stackrel{def}{=} \phi_*\Big(\mathcal{J}(\lambda)\Big), \quad \lambda \in \gamma, \tag{2.5}$$

from γ to $L(T_\gamma N)$.

2.2 Fundamental form of the curve in Lagrange Grassmannian Jacobi curves are invariants of the distribution D. They are unparametrized curves in the Lagrange Grassmannians. In [3] for any curve of so-called constant *weight* in Lagrange Grassmannian we construct *the canonical projective structure* and a special degree 4 differential, *fundamental form*, which are invariants w.r.t. the action of linear Symplectic Group $GL(W)$ and reparametrization of the curve. Below we describe briefly the construction of these invariants. Actually it is more convenient to work with the curve in the set $G_m(W)$ be the set of all m-dimensional subspaces of $2m$-dimensional linear space W (i.e., in the Grassmannian of half-dimensional subspaces), where General Linear Group acts. Since any curve of Lagrange subspaces w.r.t. some symplectic form on W is obviously the curve in $G_m(W)$, all constructions below are valid for the curves in Lagrange Grassmannian.

For given $\Lambda \in G_m(W)$ denote by $\Lambda^{\pitchfork}$ the set of all m-dimensional subspaces of W transversal to Λ,

$$\Lambda^{\pitchfork} = \{\Gamma \in G_m(W) : W = \Gamma \oplus \Lambda\} = \{\Gamma \in G_m(W) : \Gamma \cap \Lambda = 0\}$$

Fix some $\Delta \in \Lambda^{\pitchfork}$. Then for any subspace $\Gamma \in \Lambda^{\pitchfork}$ there exist unique linear mapping from Δ to Λ with graph Γ. We denote this mapping by $\langle \Delta, \Gamma, \Lambda \rangle$. So,

$$\Gamma = \{v + \langle \Delta, \Gamma, \Lambda \rangle v | v \in \Delta\}.$$

Choosing the bases in Δ and Λ one can assign to any $\Gamma \in \Lambda^{\pitchfork}$ the matrix of the mapping $\langle \Delta, \Gamma, \Lambda \rangle$ w.r.t these bases. In this way we define the coordinates on the set $\Lambda^{\pitchfork}$.

Remark 2.2 Assume that W is provided with some symplectic form $\bar{\sigma}$ and Δ, Λ are Lagrange subspaces w.r.t. $\bar{\sigma}$. Then the map $v \mapsto \bar{\sigma}(v, \cdot)$, $v \in \Delta$, defines the canonical isomorphism between Δ and Λ^*. It is easy to see that Γ is Lagrange subspace iff the mapping $\langle \Delta, \Gamma, \Lambda \rangle$, considered as the mapping from Λ^* to Λ, is self-adjoint. □

Let $\Lambda(t)$ be a smooth curve in $G_m(W)$ defined on some interval $I \subset \mathbb{R}$. We are looking for invariants of $\Lambda(t)$ by the action of $GL(W)$. We say that the curve $\Lambda(\cdot)$ is *ample at* τ if $\exists s > 0$ such that for any representative $\Lambda^s_\tau(\cdot)$ of the s-jet of $\Lambda(\cdot)$ at τ, $\exists t$ such that $\Lambda^s_\tau(t) \cap \Lambda(\tau) = 0$. The curve $\Lambda(\cdot)$ is called *ample* if it is ample at any point. This is an intrinsic definition of an ample curve. In coordinates this definition takes the following form: if in some coordinates the curve $\Lambda(\cdot)$ is a curve of matrices $t \mapsto S_t$, then $\Lambda(\cdot)$ is ample at τ if and only if the function $t \mapsto det(S_t - S_\tau)$ has a root of *finite order* at τ.

Definition 1 *The order of zero of the function* $t \mapsto det(S_t - S_\tau)$ *at* τ, *where* S_t *is a coordinate representation of the curve* $\Lambda(\cdot)$, *is called a weight of the curve* $\Lambda(\cdot)$ *at* τ.

It is clear that the weight of $\Lambda(\tau)$ is integral valued upper semicontinuous functions of τ. Therefore it is locally constant on the open dense subset of the interval of definition I.

Now suppose that the curve has the constant weight k on some subinterval $I_1 \subset I$. It implies that for all two parameters t_0,t_1 in I_1 sufficiently such that $t_0 \neq t_1$, one has

$$\Lambda(t_0) \cap \Lambda(t_1) = 0.$$

Hence for such t_0, t_1 the following linear mappings

$$\frac{d}{ds} \langle \Lambda(t_0), \Lambda(s), \Lambda(t_1) \rangle \Big|_{s=t_0} : \Lambda(t_0) \mapsto \Lambda(t_1), \tag{2.6}$$

$$\frac{d}{ds} \langle \Lambda(t_1), \Lambda(s), \Lambda(t_0) \rangle \Big|_{s=t_1} : \Lambda(t_1) \mapsto \Lambda(t_0) \tag{2.7}$$

are well defined. Taking composition of mapping (2.7) with mapping (2.6) we obtain the operator from the subspace $\Lambda(t_0)$ to itself.

Remark 2.3 This operator is actually the infinitesimal cross-ratio of two points $\Lambda(t_i)$, $i = 0,1$, together with two tangent vectors $\dot\Lambda(t_i)$, $i = 0,1$, at these points in $G_m(W)$ (see [3] for the details). □

Theorem 1 *(see [3], Lemma 4.2) If the curve has the constant weight k on some subinterval $I_1 \subset I$, then the following asymptotic holds*

$$\operatorname{tr}\Big(\tfrac{d}{ds}\langle\Lambda(t_1),\Lambda(s),\Lambda(t_0)\rangle\Big|_{s=t_1} \circ \tfrac{d}{ds}\langle\Lambda(t_0),\Lambda(s),\Lambda(t_1)\rangle\Big|_{s=t_0}\Big) = \\ -\tfrac{k}{(t_0-t_1)^2} - g_\Lambda(t_0,t_1), \tag{2.8}$$

where $g_{\Lambda}(t,\tau)$ is a smooth function in the neighborhood of diagonal $\{(t,t)|t \in I_1\}$.

The function $g_{\Lambda}(t,\tau)$ is "generating" function for invariants of the parametrized curve by the action of $GL(2m)$. The first coming invariant of the parametrized curve, *the generalized Ricci curvature*, is just $g_{\Lambda}(t,t)$, the value of g_{Λ} at the diagonal.

In order to obtain invariants for unparametrized curves (i.e., for one-dimensional submanifold of $G_m(W)$) we use a simple reparametrization rule for a function g_{Λ}. Indeed, let $\varphi : \mathbb{R} \mapsto \mathbb{R}$ be a smooth monotonic function. It follows directly from (2.8) that

$$g_{\Lambda\circ\varphi}(t_0,t_1) = \dot\varphi(t_0)\dot\varphi(t_1)g_{\Lambda}(\varphi(t_0),\varphi(t_1)) + k\left(\frac{\dot\varphi(t_0)\dot\varphi(t_1)}{(\varphi(t_0)-\varphi(t_1))^2} - \frac{1}{(t_0-t_1)^2}\right). \tag{2.9}$$

In particular, putting $t_0 = t_1 = t$, one obtains the following reparametrization rule for the generalized Ricci curvature

$$g_{\Lambda\circ\varphi}(t,t) = \dot\varphi(t)^2 g_{\Lambda}(\varphi(t),\varphi(t)) + \frac{k}{3}\mathbb{S}(\varphi), \tag{2.10}$$

where $\mathbb{S}(\varphi)$ is a Schwarzian derivative of φ,

$$\mathbb{S}(\varphi) = \frac{1}{2}\frac{\varphi^{(3)}}{\varphi'} - \frac{3}{4}\Big(\frac{\varphi''}{\varphi'}\Big)^2 = \frac{d}{dt}\Big(\frac{\varphi''}{2\varphi'}\Big) - \Big(\frac{\varphi''}{2\varphi'}\Big)^2. \tag{2.11}$$

From (2.10) it follows that the class of local parametrizations, in which the generalized Ricci curvature is identically equal to zero, defines a *canonical projective structure* on the curve (i.e., any two parametrizations from this class are transformed one to another by Möbius transformation). This parametrizations are called projective. From (2.9) it follows that if t and τ are two projective parametrizations on the curve $\Lambda(\cdot)$, $\tau = \varphi(t) = \frac{at+b}{ct+d}$, and g_Λ is generating function of $\Lambda(\cdot)$ w.r.t. parameter τ then

$$\frac{\partial^2 g_{\Lambda\circ\varphi}}{\partial t_1^2}(t_0,t_1)\Big|_{t_0=t_1=t} = \frac{\partial^2 g_{\Lambda}}{\partial \tau_1^2}(\tau_0,\tau_1)\Big|_{\tau_0=\tau_1=\varphi(t)}(\varphi'(t))^4, \tag{2.12}$$

which implies that the following degree four differential

$$\mathcal{A} = \frac{1}{2}\frac{\partial^2 g_{\Lambda}}{\partial \tau_1^2}(\tau_0,\tau_1)\Big|_{\tau_0=\tau_1=\tau}(d\tau)^4$$

on the curve $\Lambda(\cdot)$ does not depend on the choice of the projective parametrization (by degree four differential on the curve we mean the following: for any point of the curve a degree 4 homogeneous function is given on the tangent line to the curve at this point). This degree four differential is called a *fundamental form* of the curve. If t is an arbitrary (not necessarily projective) parametrization on the curve $\Lambda(\cdot)$, then the fundamental form in this parametrization has to be of the form $A(t)(dt)^4$, where $A(t)$ is a smooth function, called the *density* of the fundamental form w.r.t. parametrization t.

Now we suppose that the linear space W is provided with some symplectic form $\bar{\sigma}$ and we restrict ourselves to the curves in the corresponding Lagrange Grassmannian $L(W)$. The tangent space $T_\Lambda L(W)$ to any $\Lambda \in L(W)$ can be identified with the space of quadratic forms $Q(\Lambda)$ on the linear space Λ. Indeed, take a curve $\Lambda(t) \in G_k(W)$ with $\Lambda(0) = \Lambda$. Given some vector $l \in \Lambda$, take a curve $l(\cdot)$ in W such that $l(t) \in \Lambda(t)$ for all sufficiently small t and $l(0) = l$. It is easy to see that the quadratic form $l \mapsto \bar{\sigma}\big(l'(0), l\big)$ depends only on $\frac{d}{dt}\Lambda(0)$. In this way we identify $\frac{d}{dt}\Lambda(0) \in T_\Lambda G_k(W)$ with some element of $Q(\Lambda)$ (a simple counting of dimension shows that these correspondence between $T_\Lambda L(W)$ and $Q(\Lambda)$ is a bijection). The rank of the curve $\Lambda(t) \subset L(W)$ is by definition the rank of its velocity $\frac{d}{dt}\Lambda(t)|_{t=\tau}$ at τ, considered as the quadratic form. The curve $\Lambda(t)$ in $L(W)$ is called *nondecreasing (nonincreasing)* if its velocities $\frac{d}{dt}\Lambda(t)$ at any point are nonnegative (nonpositive) definite quadratic forms.

2.3 Construction of fundamental form of (2,5)- distributions Note that Jacobi curve J_γ of characteristic curve γ of distribution D defined by (2.5) is not ample, because all subspaces $J_\gamma(\lambda)$ have a common line. Indeed, let $\delta_a : T^*M \mapsto T^*M$ be the homothety by $a \neq 0$ in the fibers, namely,

$$\delta_a(p, q) = (ap, q), \quad q \in M, \ p \in T^*M. \tag{2.13}$$

Denote by $\vec{e}(\lambda)$ the following vector field called Euler field

$$\vec{e}(\lambda) = \frac{\partial}{\partial a}\delta_a(\lambda)\Big|_{a=1} \tag{2.14}$$

Remark 2.4 Obviously, if γ is characteristic curve of D, then also $\delta_a(\gamma)$ is. $\Box$

It implies that the vectors $\phi_*(\vec{e}(\lambda))$ coincide for all $\lambda \in \gamma$, so the line

$$E_\gamma \stackrel{def}{=} \{\mathbb{R}\phi_*(\vec{e}(\lambda))\} \tag{2.15}$$

is common for all subspaces $J_\gamma(\lambda)$, $\lambda \in \gamma$ (here, as in Introduction, $\phi : O_\gamma \to N$ is the canonical projection on the factor $N = O_\gamma/(Ab_D|_{O_\gamma})$, where O_γ is sufficiently small tubular neighborhood of γ in $(D^2)^\perp$).

Therefore it is natural to make an appropriate factorization by this common line E_γ. Namely, by above all subspaces $J_\gamma(\lambda)$ belong to skew-symmetric complement $E_\gamma^\angle$ of E_γ in $T_\gamma N$. Denote by $p : T_\gamma N \mapsto T_\gamma N/E_\gamma$ the canonical projection on the factor-space. The mapping

$$\lambda \mapsto \widetilde{J}_\gamma(\lambda) \stackrel{def}{=} p(J_\gamma(\lambda)), \ \lambda \in \gamma \tag{2.16}$$

from γ to $L(E_\gamma^\angle/E_\gamma)$ is called *reduced Jacobi curve* of characteristic curve γ. Note that in the considered case

$$\dim E_\gamma^\angle/E_\gamma = 4 \tag{2.17}$$

To clarify the notion of the reduced Jacobi curve note that for any $\lambda \in \gamma$ one can make the following identification

$$T_\gamma N \sim T_\lambda (D^2)^\perp / \ker \sigma|_{(D^2)^\perp}(\lambda). \tag{2.18}$$

Take on O_γ any vector field H tangent to characteristic 1-foliation Ab_D and without stationary points, i.e., $H(\lambda) \in \ker\sigma|_{(D^2)^\perp}(\lambda)$, $H(\lambda) \neq 0$ for all $\lambda \in O_\gamma$. Then it is not hard to see that under identification (2.18) one has

$$\widetilde{J}_\gamma(e^{tH}\lambda) = (e^{-tH})_*\Big(\mathcal{J}(e^{tH}\lambda)\Big)/\mathrm{span}(\ker \sigma|_{(D^2)^\perp}(\lambda), \vec{e}(\lambda)) \tag{2.19}$$

where e^{tH} is the flow generated by the vector field H, and the subspaces $\widetilde{J}_\gamma(e^{tH}\lambda)$ live in the symplectic space W_λ, defined as follows

$$W_\lambda = \Big(\vec{e}(\lambda)^\angle \cap T_\lambda (D^2)^\perp\Big)/\mathrm{span}\big(\ker \sigma|_{(D^2)^\perp}(\lambda), \vec{e}(\lambda)\big). \tag{2.20}$$

The following proposition follows directly from Propositions 2.2, 2.6 of [12] and relation (2.17).

Proposition 2.1 *The reduced Jacobi curve of characteristic curve of $(2,5)$-distribution with the small growth vector $(2,3,5)$ is rank 1 nondecreasing curve of the constant weight 4 in Lagrange Grassmannian of 4-dimensional linear symplectic space.*

In order to construct fundamental form we have to introduce some notations. Let X_1, X_2 be two vector fields, constituting the basis of distribution D, i.e.,

$$D(q) = \mathrm{span}(X_1(q), X_2(q)) \quad \forall q \in M. \tag{2.21}$$

Since our study is local, we can always suppose that such basis exists, restricting ourselves, if necessary, on some coordinate neighborhood instead of whole M. Given the basis X_1, X_2 one can construct a special vector field $\vec{h}_{X_1,X_2}$ tangent to characteristic 1-foliation Ab_D. For this suppose that

$$\begin{aligned} &X_3 = [X_1, X_2] \mod D, \quad X_4 = [X_1,[X_1,X_2]] = [X_1, X_3] \mod D^2, \\ &X_5 = [X_2,[X_1,X_2]] = [X_2, X_3] \mod D^2 \end{aligned} \tag{2.22}$$

Definition 2 *The tuple $\{X_i\}_{i=1}^5$, satisfying (2.21) and (2.22) is called adapted frame of the distribution D. If instead of (2.22) one has*

$$X_3 = [X_1, X_2], \ X_4 = [X_1,[X_1,X_2]] = [X_1,X_3], \ X_5 = [X_2,[X_2,X_1]] = [X_3, X_2], \tag{2.23}$$

the frame $\{X_i\}_{i=1}^5$ will be called strongly adapted to D.

Let us introduce "quasi-impulses" $u_i : T^*M \mapsto \mathbb{R}$, $1 \leq i \leq 5$,

$$u_i(\lambda) = p \cdot X_i(q), \ \lambda = (p,q), \ q \in M, \ p \in T_q^*M \tag{2.24}$$

For given function $G : T^*M \mapsto \mathbb{R}$ denote by $\vec{G}$ the corresponding Hamiltonian vector field defined by the relation $\sigma(\vec{G}, \cdot) = d\,G(\cdot)$. Then it is easy to show (see, for example [11]) that

$$\ker \sigma\Big|_{D^\perp}(\lambda) = \mathrm{span}(\vec{u}_1(\lambda), \vec{u}_2(\lambda)), \quad \forall \lambda \in D^\perp, \tag{2.25}$$

$$\ker\sigma\Big|_{(D^2)^\perp}(\lambda)=\mathbb{R}\Big((u_4\vec u_2-u_5\vec u_1)(\lambda)\Big),\quad \forall\lambda\in(D^2)^\perp\backslash(D^3)^\perp \tag{2.26}$$

The last relation implies that the following vector field

$$\vec h_{X_1,X_2}=u_4\vec u_2-u_5\vec u_1 \tag{2.27}$$

is tangent to the characteristic 1-foliation.

Now we are ready to construct the fundamental form. For any $\lambda\in(D^2)^\perp\backslash(D^3)^\perp$ take characteristic curve γ, passing through λ. Let $\mathcal{A}_\lambda$ be the fundamental form of the reduced Jacobi curve $\widetilde J_\gamma(\lambda)$ of γ at λ (by Proposition 2.1 $\mathcal{A}_\lambda$ is well defined). By construction $\mathcal{A}_\lambda$ is degree 4 homogeneous function on the tangent line to γ at λ. Given a local basis (X_1,X_2) of distribution D let

$$A_{X_1,X_2}(\lambda)=\mathcal{A}_\lambda(\vec h_{X_1,X_2}(\lambda)) \tag{2.28}$$

In this way to any (local) basis (X_1,X_2) of distribution D we assign the function A_{X_1,X_2} on $(D^2)^\perp\backslash(D^3)^\perp$.

Remark 2.5 If we consider parametrization $t\mapsto\widetilde J_\gamma(e^{t\vec h_{X_1,X_2}}\lambda)$ of the reduced Jacobi curve of γ, then $A_{X_1,X_2}(\lambda)$ is the density of fundamental form of this curve w.r.t. parametrization t at $t=0$. $\Box$

Take another basis $\tilde X_1,\tilde X_2$ of the distribution D. It can be shown (see [12], beginning of subsection 2.3) that for any $q\in M$ the restriction of the corresponding function $A_{\tilde X_1,\tilde X_2}$ to the fiber $(D^2)^\perp(q)\backslash(D^3)^\perp(q)$ $(=(D^2)^\perp(q)\backslash\{0\}$ by Remark 2.1) is equal to the restriction of A_{X_1,X_2} to $(D^2)^\perp(q)\backslash\{0\}$, multiplied on some positive constant (the square of the determinant of transition matrix from the basis (X_1,X_2) to the basis $(\tilde X_1,\tilde X_2)$). The following Proposition is direct consequence of the last assertion, Proposition 2.8 and Theorem 3 from [12]:

Proposition 2.2 *If D is $(2,5)$ distribution on M with the small growth vector $(2,3,5)$ at any point, then the restriction of A_{X_1,X_2} to the fiber $(D^2)^\perp(q)$ is well defined degree 4 homogeneous polynomial, up to multiplication on positive constant.*

Definition 3 *The restriction of A_{X_1,X_2} to $(D^2)^\perp(q)$ will be called fundamental form of $(2,5)$-distribution D at the point q.*

From now on we will write $\vec h$ instead of $\vec h_{X_1,X_2}$ and A instead of A_{X_1,X_2} without special mentioning.

In the case $n=5$ and small growth vector $(2,3,5)$ one can look at the fundamental form of the distribution D from the different point of view. In this case (in contrast to generic $(2,n)$-distributions with $n>5$) there is only one abnormal trajectory starting at given point $q\in M$ in given direction (tangent to $D(q)$). All lifts of this abnormal trajectory can be obtained one from another by homothety. So they have the same, up to symplectic transformation, Jacobi curve. It means that one can consider Jacobi curve and fundamental form of this curve on abnormal trajectory instead of abnormal extremal. Therefore, to any $q\in M$ one can assign a homogeneous degree 4 polynomial $\mathring{A}_q$ on the plane $D(q)$ in the following way:

$$\mathring{A}_q(v)\overset{def}{=}\mathcal{A}_\lambda(H) \tag{2.29}$$

for any $v \in D(q)$, where

$$\pi(\lambda) = q, \quad \pi_* H = v, \quad H \in \ker \sigma|_{(D^2)^\perp}(\lambda). \tag{2.30}$$

and the righthand side of (2.29) is the same for any choice of λ and H, satisfying (2.30).

$\mathring{A}_q$ will be called *tangential fundamental form* of the distribution D at the point q. We stress that the tangential fundamental form is the well defined degree 4 homogeneous polynomial on $D(q)$ and not the class of polynomials defined up to multiplication on a positive constant.

In [12] in the case $n = 5$ we obtained the explicit formulas for calculation of the (tangential) fundamental form (see Theorem 2 there), but they are not sufficient for our purposes here.

3 Cartan's tensor

In the present section, following chapter VI of the original paper [7], we briefly describe the main steps of construction of Cartan's tensor, rewriting all formulas that we need for the comparison of our and Cartan's invariants. We will use the language of Cartan and his notations, referring sometimes by remarks to modern terminology of G-structures (see [8],[9],[10]). In order to simplify the formulas, we will omit in the sequel the sign $\wedge$ for standard operation with differential forms.

Let ω_1, ω_2, ω_3, ω_4, and ω_5 be coframe on M (i.e., $\{\omega_i(q)\}_{i=1}^5$ constitute the basis of T_q^*M for any $q \in M$) such that the rank 2 distribution D is annihilator of the first three elements of this coframe, namely,

$$D(q) = \{v \in T_qM;\ \omega_1(v) = \omega_2(v) = \omega_3(v) = 0\}, \quad \forall q \in M \tag{3.1}$$

Obviously the set of all coframes satisfying (3.1) is 19-parametric family. Among all these coframes Cartan distinguishes special coframes satisfying the following structural equations (formula (VI.5) in [7]):

$$\begin{aligned}
d\,\omega_1 &= \omega_1(2\overline{\omega}_1 + \overline{\omega}_4) + \omega_2\overline{\omega}_2 + \omega_3\omega_4\\
d\,\omega_2 &= \omega_1\overline{\omega}_3 + \omega_2(\overline{\omega}_1 + 2\overline{\omega}_4) + \omega_3\omega_5\\
d\,\omega_3 &= \omega_1\overline{\omega}_5 + \omega_2\overline{\omega}_6 + \omega_3(\overline{\omega}_1 + \overline{\omega}_4) + \omega_4\omega_5\\
d\,\omega_4 &= \omega_1\overline{\omega}_7 + \frac{4}{3}\omega_3\overline{\omega}_6 + \omega_4\overline{\omega}_1 + \omega_5\overline{\omega}_2\\
d\,\omega_5 &= \omega_2\overline{\omega}_7 - \frac{4}{3}\omega_3\overline{\omega}_5 + \omega_4\overline{\omega}_3 + \omega_5\overline{\omega}_4,
\end{aligned} \tag{3.2}$$

where $\overline{\omega}_j$, $1 \le j \le 7$, are new 1-forms. It turns out that the set of all coframes satisfying (3.2) is 7-parametric family. This family defines 12-dimensional bundle over M , which will be denoted by $\mathcal{K}(M)$: instead of considering a family of coframes $\{\omega_i\}_{i=1}^5$ on M one can consider the 5-tuple of 1-forms on $\mathcal{K}(M)$; one can think of the forms $\overline{\omega}_j$, $1 \le j \le 7$, from (3.2) and of the equation (3.2) itself as defined on $\mathcal{K}(M)$.

Remark 3.1 In the modern terminology $\mathcal{K}(M)$ is G-structure over M such that the Lie algebra of its structure group is the algebra of the following matrices

$$\begin{pmatrix} 2a_1+a_4 & a_2 & 0 & 0 & 0 \\ a_3 & a_1+2a_4 & 0 & 0 & 0 \\ a_5 & a_6 & a_1+a_4 & 0 & 0 \\ a_7 & 0 & \frac{4}{3}a_6 & a_1 & a_2 \\ 0 & a_7 & -\frac{4}{3}a_5 & a_3 & a_4 \end{pmatrix} \tag{3.3}$$

(compare this with (3.2)).One says that the G-structure $\mathcal{K}(M)$ is obtained by reduction procedure from the bundle of all coframes satisfying (3.1). It turns out that the further reduction is impossible. □

Note also that the tuple of the forms $\left(\{\omega_i\}_{i=1}^5, \{\overline{\omega}_j\}_{j=1}^7\right)$ is coframe on $\mathcal{K}(M)$. Further, it turns out that the tuple of the forms $\overline{\omega}_j$, $1 \le j \le 7$, is defined by structure equation (3.2) up to the following transformations (the formula after the formula (VI.7) in [7]):

$$\begin{gathered} \left(\overline{\omega}_1, \overline{\omega}_2, \overline{\omega}_3, \overline{\omega}_4, \overline{\omega}_5, \overline{\omega}_6, \overline{\omega}_7\right) \to \left(\overline{\omega}_1 + \nu_1\omega_1, \overline{\omega}_2 + \nu_2\omega_1, \right. \\ \left. \overline{\omega}_3 + \nu_1\omega_2, \overline{\omega}_4 + \nu_2\omega_2, \overline{\omega}_5 + \nu_1\omega_3, \overline{\omega}_6 + \nu_2\omega_3, \overline{\omega}_7 + \nu_1\omega_4 + \nu_2\overline{\omega}_5\right), \end{gathered} \tag{3.4}$$

where parameters ν_1 and ν_2 are arbitrary. Replacing the tuple $\overline{\omega}_j$, $1 \le j \le 7$, by the righthand side of (3.4), one has 2-parametric family of coframes $\left(\{\omega_i\}_{i=1}^5, \{\overline{\omega}_j\}_{j=1}^7\right)$ on $\mathcal{K}(M)$. This family defines 14-dimensional bundle over $\mathcal{K}(M)$, which will be denoted by $\mathcal{K}_1(M)$.

Remark 3.2 One says that the bundle $\mathcal{K}_1(M)$ is prolongation of $\mathcal{K}(M)$.□

Further Cartan expresses seven exterior derivatives $d\overline{\omega}$, considered as the forms on $\mathcal{K}_1(M)$, by the 1-forms $\{\omega_i\}_{=1}^5$, $\{\overline{\omega}_j\}_{j=1}^7$ and two new forms χ_1 and χ_2 on $\mathcal{K}_1(M)$ such that these two new forms are defined uniquely by these expressions(see formula (VI.8) in [7]). Therefore the tuple

$$\left(\{\omega_i\}_{i=1}^5, \{\overline{\omega}_j\}_{j=1}^7, \chi_1, \chi_2\right) \tag{3.5}$$

is canonical coframe on $\mathcal{K}_1(M)$.

Remark 3.3 The pair $\mathcal{K}_1(M)$ and coframe (3.5) can be considered as the trivial bundle $\mathcal{K}_2(M)$ over $\mathcal{K}_1(M)$, which is actually one more prolongation. So, by reduction from the bundle of coframes satisfying (3.1) to the bundle $\mathcal{K}(M)$ and by two successive prolongations (from $\mathcal{K}(M)$ to $\mathcal{K}_1(M)$ and then to $\mathcal{K}_2(M)$) one can arrive to the canonical coframe, which essentially solve the problem of equivalence of the considered class of distributions. □

In the sequel we will need the following consequences of the formula (VI.8) in [7]:

$$\begin{aligned} d\left(\overline{\omega}_1 - \overline{\omega}_4\right) &= 2\overline{\omega}_3\overline{\omega}_2 - \omega_4\overline{\omega}_5 + \omega_5\overline{\omega}_6 + \omega_1\chi_1 - \omega_2\chi_2 + 3B_2\omega_1\omega_3 + \\ &\quad 3B_3\omega_2\omega_3 + 3A_2\omega_1\omega_4 + 3A_3\omega_1\omega_5 + 3A_3\omega_2\omega_4 + 3A_4\omega_2\omega_5, \qquad (3.6) \\ d\overline{\omega}_2 &= \overline{\omega}_2(\overline{\omega}_1 - \overline{\omega}_4) - \omega_4\overline{\omega}_6 + \omega_1\chi_2 + B_4\omega_2\omega_3 + A_4\omega_2\omega_4 + A_5\omega_2\omega_5, \qquad (3.7) \\ d\overline{\omega}_3 &= \overline{\omega}_3(\overline{\omega}_4 - \overline{\omega}_1) - \omega_5\overline{\omega}_5 + \omega_2\chi_1 - B_1\omega_1\omega_3 - A_1\omega_1\omega_4 - A_2\omega_1\omega_5, \qquad (3.8) \end{aligned}$$

Here A's and B's are functions on $\mathcal{K}_1(M)$.

Cartan considers the following expression

$$\mathcal{F} = A_1\omega_4^4 + 4A_2\omega_4^3\omega_5 + 6A_3\omega_4^2\omega_5^2 + 4A_4\omega_4\omega_5^3 + A_5\omega_5^4. \tag{3.9}$$

Let $\mathcal{P} : \mathcal{K}_1(M) \mapsto M$ be the canonical projection. Note that for given $Q \in \mathcal{K}_1(M)$ the form $\mathcal{F}$ at Q can be considered as degree 4 homogeneous polynomial on $T_{\mathcal{P}(Q)}M$. It turns out that the form $\mathcal{F}$ calculated at different points of the same fiber $\mathcal{P}^{-1}(q)$, $q \in M$, gives the same polynomial (modulo $\omega_1(q)$, $\omega_2(q)$, $\omega_3(q)$) on T_qM or, equivalently, the same polynomial on the plane $D(q)$ (recall that by construction $D(q)$ is annihilator of $\mathrm{span}(\omega_1(q), \omega_2(q), \omega_3(q))$. Briefly speaking, $\mathcal{F}$ restricted on D is covariant symmetric tensor of order 4. This tensor is called *Cartan's tensor* of distribution D. We will denote by $\mathcal{F}_q$ Cartan's tensor at the point q.

So for any $q \in M$ our tangential fundamental form $\mathring{A}_q$ and Cartan's tensor $\mathcal{F}_q$ are both invariantly defined degree 4 homogeneous polynomials on the plane $D(q)$. The following theorem is the main results of the present paper:

Theorem 2 *For any $q \in M$ Cartan's tensor $\mathcal{F}_q$ and the tangential fundamental form $\mathring{A}_q$ are connected by the following identity*

$$\mathcal{F}_q = -35\mathring{A}_q. \tag{3.10}$$

As a conclusion of this theorem we obtain that geometric reason for the existence of Cartan's tensor is the existence of the special degree 4 differential on curves in Grassmannian of half-dimensional subspaces, which is constructed with the help of the notion of the cross-ratio of four point in this Grassmanian (or of the infinitesimal cross-ratio of two points in this Grassmannian together with two tangent vectors at these points).

4 Fundamental form for $n = 5$ in terms of the structural functions of some adapted frame

In the present section we make preparations to prove Theorem 2. Namely, for given adapted frame to the distribution we derive the formula for fundamental form in terms of its structural functions. This formula is important not only in the proof of this theorem: in many cases it is more efficient from computational point of view than the method given in [12] (see Theorem 2 there).

First we need more facts from the theory of curves in Grassmannian $G_mW)$ of half-dimensional subspaces (here $\dim W = 2m$) and in Lagrange Grassmannian $L(W)$ w.r.t. to some symplectic form on W, developed in [3], [4]). Below we present all necessary facts from the mentioned papers together with several new useful arguments.

Fix some $\Lambda \in G_m(W)$. As before , let $\Lambda^\pitchfork$ be the set of all m-dimensional subspaces of W transversal to Λ. Note that any $\Delta \in \Lambda^\pitchfork$ can be canonically identified with W/Λ. Keeping in mind this identification and taking another subspace $\Gamma \in \Lambda^\pitchfork$ one can define the operation of subtraction $\Gamma - \Delta$ as follows

$$\Gamma - \Delta \stackrel{def}{=} \langle \Delta, \Gamma, \Lambda \rangle \in \mathrm{Hom}\,(W/\Lambda, \Lambda).$$

It is clear that the set $\Lambda^\pitchfork$ provided with this operation can be considered as the affine space over the linear space $\mathrm{Hom}\,(W/\Lambda, \Lambda)$.

Consider now some ample curve $\Lambda(\cdot)$ in $G_m(W)$. Fix some parameter τ. By assumptions $\Lambda(t) \in \Lambda(\tau)^\pitchfork$ for all t from a punctured neighborhood of τ. We obtain the curve $t \mapsto \Lambda(t) \in \Lambda(\tau)^\pitchfork$ in the affine space $\Lambda(\tau)^\pitchfork$ with the pole at τ. We denote by $\Lambda_\tau(t)$ the identical embedding of $\Lambda(t)$ in the affine space $\Lambda(\tau)^\pitchfork$. First note that the velocity $\frac{\partial}{\partial t}\Lambda_\tau(t)$ is well defined element of

$\mathrm{Hom}(W/\Lambda,\Lambda)$. Fixing an "origin" in $\Lambda(\tau)^{\pitchfork}$ we make $\Lambda_\tau(t)$ a vector function with values in $\mathrm{Hom}\,(W/\Lambda,\Lambda)$ and with the pole at $t=\tau$. Obviously, only free term in the expansion of this function to the Laurent series at τ depends on the choice of the "origin" we did to identify the affine space with the linear one. More precisely, the addition of a vector to the "origin" results in the addition of the same vector to the free term in the Laurent expansion. In other words, for the Laurent expansion of a curve in an affine space, the free term of the expansion is an element of this affine space. Denote this element by $\Lambda^0(\tau)$. The curve $\tau\mapsto\Lambda^0(\tau)$ is called the *derivative curve* of $\Lambda(\cdot)$.

If we restrict ourselves to the Lagrange Grassmannian $L(W)$, i.e. if all subspaces under consideration are Lagrangian w.r.t. some symplectic form $\bar\sigma$ on W, then from Remark 2.2 it follows easily that the set $\Lambda_L^{\pitchfork}$ of all Lagrange subspaces transversal to Λ can be considered as the affine space over the linear space of all self-adjoint mappings from Λ^* to Λ, the velocity $\frac{\partial}{\partial t}\Lambda_\tau(t)$ is well defined self-adjoint mappings from Λ^* to Λ, and the derivative curve $\Lambda^0(\cdot)$ consist of Lagrange subspaces. Besides if the curve $\Lambda(\cdot)$ is nondecreasing rank 1 curve in $L(W)$, then $\frac{\partial}{\partial t}\Lambda_\tau(t)$ is a nonpositve definite rank 1 self-adjoint map from Λ^* to Λ and for $t\neq\tau$ there exists a unique, up to the sign, vector $w(t,\tau)\in\Lambda(\tau)$ such that for any $v\in\Lambda(\tau)^*$

$$\langle v,\frac{\partial}{\partial t}\Lambda_\tau(t)v\rangle=-\langle v,w(t,\tau)\rangle^2. \tag{4.1}$$

The properties of vector function $t\mapsto w(t,\tau)$ for a rank 1 curve of constant weight in $L(W)$ can be summarized as follows (see [3], section 7, Proposition 4 and Corollary 2):

Proposition 4.1 *If $\Lambda(\cdot)$ is a rank 1 curve of constant weight in $L(W)$, then for any $\tau\in I$ the function $t\mapsto w(t,\tau)$ has a pole of order m at $t=\tau$. Moreover, if we write down the expansion of $t\mapsto w(t,\tau)$ in Laurent series at $t=\tau$,*

$$w(t,\tau)=\sum_{i=1}^m e_i(\tau)(t-\tau)^{i-1-l}+O(1),$$

then the vector coefficients $e_1(\tau),\dots,e_m(\tau)$ constitute a basis of the subspace $\Lambda(t)$.

The basis of vectors $e_1(\tau),\dots,e_m(\tau)$ from the previous proposition is called a *canonical basis* of $\Lambda(\tau)$. Further for given τ take the derivative subspace $\Lambda^0(\tau)$ and let $f_1(\tau),\dots,f_m(\tau)$ be a basis of $\Lambda^0(\tau)$ dual to the canonical basis of $\Lambda(\tau)$, i.e. $\bar\sigma(f_i(\tau),e_j(\tau))=\delta_{i,j}$. The basis

$$(e_1(\tau),\dots,e_m(\tau),f_1(\tau),\dots,f_m(\tau))$$

of whole symplectic space W is called *the canonical moving frame* of the curve $\Lambda(\cdot)$. Calculation of structural equation for the canonical moving frame is another way to obtain symplectic invariants of the curve $\Lambda(\cdot)$.

For the reduced Jacobi curves of abnormal extremals of $(2,5)$-distribution m is equal to 2. So we restrict ourselves to this case. For $m=2$ the structural equation for the canonical moving frame has the following form (for the proof see [4] Section 2, Proposition 7):

$$\begin{pmatrix} e_1'(\tau)\\ e_2'(\tau)\\ f_1'(\tau)\\ f_2'(\tau)\end{pmatrix}=\begin{pmatrix} 0 & 3 & 0 & 0\\ \frac14\rho(\tau) & 0 & 0 & 4\\ -\left(\frac{35}{36}A(\tau)-\frac18\rho(\tau)^2+\frac1{16}\rho''(\tau)\right) & -\frac7{16}\rho'(\tau) & 0 & -\frac14\rho(\tau)\\ -\frac7{16}\rho'(\tau) & -\frac94\rho(\tau) & -3 & 0\end{pmatrix}\begin{pmatrix} e_1(\tau)\\ e_2(\tau)\\ f_1(\tau)\\ f_2(\tau)\end{pmatrix}, \tag{4.2}$$

where $\rho(\tau)$ and $A(\tau)$ are the Ricci curvature and the density of fundamental form of the parametrized curve $\Lambda(\tau)$ respectively.

Here we use the method of computation, which is slightly different from the method of section 3 of [12], using again the structural equation (4.2). Vector e_1 (and therefore e_2) can be found relatively easy (see Proposition 3.3 in [12]). On the other hand, it is difficult to compute the vectors f_1 and f_2, because they are defined with the help of the derivative curve. Instead of this one can complete the canonical basis $\big(e_1, e_2\big)$ of the curve somehow to the symplectic moving frame, find the structural equation for this frame and express the invariants ρ and A in terms of this structural equation.

Indeed, let $\tilde{f}_1(t), \tilde{f}_2(t)$ be another two vectors which complete the basis $\big(e_1(t), e_2(t)\big)$ to a symplectic moving frame in W. Then it is easy to show (see [3], section 7) that the structural equation for the frame $\big(e_1(t), e_2(t), \tilde{f}_1(t), \tilde{f}_2(t)\big)$ has always the form

$$\begin{pmatrix} e_1'(t) \\ e_2'(t) \\ \tilde{f}_1'(t) \\ \tilde{f}_2'(t) \end{pmatrix} = \begin{pmatrix} 0 & 3 & 0 & 0 \\ a_{21}(t) & a_{22}(t) & 0 & 4 \\ a_{31}(t) & a_{41}(t) & 0 & -a_{21}(t) \\ a_{41}(t) & a_{42}(t) & -3 & -\alpha_{22}(t) \end{pmatrix} \begin{pmatrix} e_1(t) \\ e_2(t) \\ \tilde{f}_1(t) \\ \tilde{f}_2(\tau) \end{pmatrix}. \tag{4.3}$$

The following lemma is the base for our method:

Lemma 4.1 *The Ricci curvature $\rho(t)$ and the density $A(t)$ of the fundamental form of the curve $\Lambda(t)$ satisfy*

$$\rho = -\frac{4}{15}\Big(3a_{21} + 2a_{42} + \frac{1}{2}\alpha_{22}' + \frac{1}{2}\alpha_{22}^2\Big), \tag{4.4}$$

$$A = \frac{36}{35}\Big(-a_{31} + \frac{9}{64}\rho^2 + \frac{1}{16}\rho'' - \frac{1}{4}(a_{21})^2 + \frac{1}{3}a_{41}' + \frac{1}{12}a_{21}'' + \frac{1}{12}(a_{21}a_{22})'\Big). \tag{4.5}$$

Proof. Obviously, the vectors $\tilde{f}_1(t), \tilde{f}_2(t)$ can be expressed by the canonical moving frame of $\Lambda(t)$ as follows

$$\begin{aligned} \tilde{f}_1 &= f_1 + \mu_{11}e_1 + \mu_{12}e_2 \\ \tilde{f}_2 &= f_2 + \mu_{12}e_1 + \mu_{22}e_2 \end{aligned} \tag{4.6}$$

It easily implies that

$$\dot{e}_2 = (a_{21} + 4\mu_{12})e_1 + (a_{22} + 4\mu_{22})e_2 + 4f_2 \tag{4.7}$$

Comparing this with the second row in (4.2) we have

$$\mu_{22} = -\frac{1}{4}a_{22} \tag{4.8}$$

$$a_{21} + 4\mu_{12} = \frac{1}{4}\rho \tag{4.9}$$

Further, using (4.8), it is easy to obtain that

$$\dot{f}_2 = (a_{41} + \frac{1}{4}a_{21}a_{22} - 3\mu_{11} - \mu_{12}')e_1 + (a_{42} + \frac{1}{4}(a_{22}^2 + a_{22}') - 6\mu_{12})e_2 - 3f_1 \tag{4.10}$$

Comparing this with the forth row in (4.2) we have

$$a_{42} + \frac{1}{4}(a_{22}^2 + a_{22}') - 6\mu_{12} = -\frac{9}{4}\rho \tag{4.11}$$

$$a_{41} + \frac{1}{4}a_{21}a_{22} - 3\mu_{11} - \mu_{12}' = -\frac{7}{16}\rho' \tag{4.12}$$

From equations (4.9) and (4.11) we obtain (4.4).

Further, from (4.9) and (4.12) it follows easily that

$$\mu_{11}=\frac{1}{3}a_{41}+\frac{1}{12}(a_{21}'+a_{21}a_{22})+\frac{1}{8}\rho' \tag{4.13}$$

Besides, by direct calculations one can get without difficulties that

$$\dot{f}_1=\Big(a_{31}-(\frac{1}{4}\rho+a_{21})\mu_{12}-\mu_{11}'\Big)e_1-\frac{7}{16}\rho' e_2-\frac{\rho}{4}f_2 \tag{4.14}$$

Comparing this with the third equation in (4.2) and using (4.9) we get

$$-\frac{35}{36}A+\frac{1}{8}\rho^2-\frac{1}{16}\rho''=a_{31}-\frac{1}{64}\rho^2+\frac{1}{4}a_{21}^2-\mu_{11}' \tag{4.15}$$

Substituting (4.13) in (4.15) one can easily obtain (4.5). □

Let us apply the previous lemma to the calculation of the fundamnetal form of (2,5)-distribution D. Let again $\{X_i\}_{i=1}^5$ be an adapted frame to D and $\vec{h}=\vec{h}_{X_1,X_2}$ be as in (2.27). Then it is not difficult to show that

$$\begin{aligned}\vec{h}=u_4\vec{u}_2-u_5\vec{u}_1=u_4X_2-u_5X_1+\Big(c_{42}^4u_4^2+(c_{42}^5-c_{41}^4)u_4u_5-c_{41}^5u_5^2\Big)\partial_{u_4}+\\ +\Big(c_{52}^4u_4^2+(c_{52}^5-c_{51}^4)u_4u_5-c_{51}^5u_5^2\Big)\partial_{u_5},\end{aligned} \tag{4.16}$$

where c_{ji}^k are the structural functions of the frame $\{X_i\}_{i=1}^5$, i.e., the functions, satisfying $[X_i,X_j]=\sum_{k=1}^5 c_{ji}^kX_k$.

For any $\lambda\in(D^2)^\perp\backslash(D^3)^\perp$ consider the characteristic curve γ of D passing through λ. Under identification (2.18) the reduced Jacobi curve $\widetilde{J}_\gamma$ lives in Lagrange Grassmannian $L(W_\lambda)$ of symplectic space W_λ, defined by (2.20). Let $\epsilon_1(\lambda)$ be the first vector in the canonical basis of the curve $t\mapsto J_\gamma(e^{t\vec{h}}\lambda)$ at the point $t=0$. Note that it is more convenient to work directly with vector fields of $(D^2)^\perp$, keeping in mind that the symplectic space W_λ belongs to the factor space $T_\lambda\big((D^2)^\perp\big)/\mathrm{span}\big(\vec{h}(\lambda),\vec{e}(\lambda)\big)$. So, in the sequel by $\epsilon_1(\lambda)$ we will mean both the element of W_λ and some representative of this element in $T_\lambda\big((D^2)^\perp\big)$, depending smoothly on λ. In the last case all equalities, containing $\epsilon_1(\lambda)$, will be assumed modulo $\mathrm{span}\big(\vec{h}(\lambda),\vec{e}(\lambda)\big)$. By Proposition 3.4 of [12] the vector $\epsilon_1(\lambda)$ can be chosen in the form

$$\epsilon_1(\lambda)=6\Big(\gamma_4(\lambda)\partial_{u_4}+\gamma_5(\lambda)\partial_{u_5}\Big), \tag{4.17}$$

where

$$\gamma_4(\lambda)u_5-\gamma_5(\lambda)u_4\equiv 1. \tag{4.18}$$

Denote by $\epsilon_2(\lambda)$ the second vector in the canonical basis of the curve $t\mapsto J_\gamma(e^{t\vec{h}}\lambda)$. Let us compute the vector $\epsilon_2(\lambda)$. First note that by definition of vector fields ϵ_1 ,$\vec{e}$ and relation (4.18) one has

$$\partial_{u_4}=\frac{u_5}{6}\epsilon_1(\mathrm{mod}\ \vec{e}),\ \partial_{u_5}=-\frac{u_4}{6}\epsilon_1(\mathrm{mod}\ \vec{e}) \tag{4.19}$$

This together with (4.16) yields easily that

$$[\vec{h},\partial_{u_i}]=\frac{1}{6}\alpha_i\epsilon_1\quad \mathrm{mod}\Big(\mathrm{span}(\vec{e},\partial_{u_1},\partial_{u_2},\partial_{u_3})\Big),\qquad i=1,2,3, \tag{4.20}$$

$$[\vec{h},\partial_{u_4}]=-\vec{u}_2+\frac{1}{6}\alpha_4\epsilon_1\quad \mathrm{mod}\Big(\mathrm{span}(\vec{e},\partial_{u_1},\partial_{u_2},\partial_{u_3})\Big), \tag{4.21}$$

$$[\vec{h},\partial_{u_5}]=\vec{u}_1+\frac{1}{6}\alpha_5\epsilon_1\quad \mathrm{mod}\Big(\mathrm{span}(\vec{e},\partial_{u_1},\partial_{u_2},\partial_{u_3})\Big), \tag{4.22}$$

where

$$\alpha_i = c^i_{52}u_4^2 - (c^i_{42} + c^i_{51})u_4u_5 + c^i_{41}u_5^2 \tag{4.23}$$

From the structural equation (4.2), relation (4.17), (4.18), (4.21),(4.22) one can easily obtain that

$$\epsilon_2 = \frac{1}{3}[\vec{h}, \epsilon_1] = 2[\vec{h}, \gamma_4\partial_{u_4} + \gamma_5\partial_{u_5}] = 2\,X - b\,\epsilon_1 \tag{4.24}$$

where

$$X = \gamma_5\vec{u}_1 - \gamma_4\vec{u}_2 + \partial_{u_3} \tag{4.25}$$

$$b = -\frac{1}{3}\Big(\vec{h}(\gamma_4)u_5 - \vec{h}(\gamma_5)u_4 + \alpha_4\gamma_4 + \alpha_5\gamma_5\Big) \tag{4.26}$$

In particular from (4.24),(4.25), and (4.18) it follows that

$$\begin{aligned} \vec{u}_1 &= -\frac{1}{2}(b\epsilon_1 + \epsilon_2)u_4 \text{ mod } \Big(\text{span}(\vec{h}, \partial_{u_3})\Big) \\ \vec{u}_2 &= -\frac{1}{2}(b\epsilon_1 + \epsilon_2)u_5 \text{ mod } \Big(\text{span}(\vec{h}, \partial_{u_3})\Big) \end{aligned} \tag{4.27}$$

Let us analyze more carefully the expression for b. From (4.18) it follows that

$$\vec{h}(\gamma_4)u_5 - \vec{h}(\gamma_5)u_4 = -\gamma_4\vec{h}(u_5) + \gamma_5\vec{h}(u_4)$$

Therefore

$$b = -\frac{1}{3}\Big(\gamma_4\big(\alpha_4 - \vec{h}(u_5)\big) + \gamma_5\big(\alpha_5 + \vec{h}(u_4)\big)\Big) \tag{4.28}$$

On the other hand, using (4.16) and (4.23), one has

$$\begin{aligned} \alpha_5 + \vec{h}(u_4) &= \Big((c^4_{42} + c^5_{52})u_4 - (c^4_{41} + c^5_{51})u_5\Big)u_4 \\ \alpha_4 - \vec{h}(u_5) &= -\Big((c^4_{42} + c^5_{52})u_4 - (c^4_{41} + c^5_{51})u_5\Big)u_5 \end{aligned} \tag{4.29}$$

Substituting the last formulas into (4.28) and using again (4.18) one has

$$b = \frac{1}{3}\Big((c^4_{42} + c^5_{52})u_4 - (c^4_{41} + c^5_{51})u_5\Big) \tag{4.30}$$

So, the function b actually does not depend on the choice of γ_4 and γ_5 and it is linear in u_4 and u_5.

Further, one can complete the canonical basis $(\epsilon_1(\lambda), \epsilon_2(\lambda))$ of the subspace $J_\gamma(\lambda)$ to the symplectic basis in the symplectic space W_λ by adding two vectors $\widetilde{\Phi}_1(\lambda)$ and $\widetilde{\Phi}_2(\lambda)$ such that

$$\widetilde{\Phi}_2 = \frac{1}{2}(\vec{u}_3 + u_4\partial_{u_1} + u_5\partial_{u_2}), \tag{4.31}$$

$$\widetilde{\Phi}_1 = \frac{1}{6}Y_4 + b\widetilde{\Phi}_2, \tag{4.32}$$

where

$$Y_4 = \frac{1}{6}\Big(u_5\vec{u}_4 - u_4\vec{u}_5 + \sum_{i=1}^{3}\big(u_5\{u_i, u_4\} - u_4\{u_i, u_5\}\big)\partial_{u_i}\Big). \tag{4.33}$$

Then the tuple

$$\big(e_1(t), e_2(t), \tilde{f}_1(t), \tilde{f}_2(t)\big) \stackrel{def}{=} (e^{-t\vec{h}})_*\big(\epsilon_1, \epsilon_2, \widetilde{\Phi}_1, \widetilde{\Phi}_2\big)(e^{t\vec{h}}\lambda) \tag{4.34}$$

is a symplectic frame in W_λ such that $(e_1(t), e_2(t))$ is canonical basis of the reduced Jacobi curve $J_\gamma(e^{t\vec{h}}\lambda)$ (in the righthand side of (4.34) one applies $(e^{-t\vec{h}})_*$ to each vector field in the brackets at the indicated point). In the sequel we will denote by $a_{ij}(\lambda)$ the entries of the matrix in the structural equation for the frame (4.34) at $t = 0$ by analogy with the equation (4.3). Namely, let

$$\begin{aligned} \dot{\tilde{f}}_1(0) &= [\vec{h}, \widetilde{\Phi}_1](\lambda) = a_{31}\epsilon_1 + a_{41}\epsilon_2 - a_{21}\widetilde{\Phi}_2 \\ \dot{\tilde{f}}_2(0) &= [\vec{h}, \widetilde{\Phi}_2](\lambda) = a_{41}\epsilon_1 + a_{42}\epsilon_2 - 3\widetilde{\Phi}_1 - a_{22}\widetilde{\Phi}_2 \end{aligned} \tag{4.35}$$

Let us calculate coefficient a_{ij},appearing in (4.35), in order to apply Lemma 4.1.

Lemma 4.2 *The following relations hold*

$$a_{22} = -b_1 - 3b, \tag{4.36}$$

$$a_{42} = -\frac{1}{4}\Pi \tag{4.37}$$

$$a_{41} = -\frac{1}{4}\Pi b + \frac{1}{12}(\alpha_1 u_4 + \alpha_2 u_5). \tag{4.38}$$

where

$$\Pi = (c_{32}^2 u_4 - c_{31}^2 u_5)u_5 - (c_{32}^1 u_4 - c_{31}^1 u_5)u_4 - \Big(u_5\{u_3, u_4\} - u_4\{u_3, u_5\}\Big) =$$

$$(c_{32}^1 + c_{53}^4)u_4^2 + (c_{32}^2 - c_{31}^1 - c_{43}^4 + c_{53}^5)u_4u_5 - (c_{31}^2 + c_{43}^5)u_5^2, \tag{4.39}$$

$$b_1 = c_{32}^3 u_4 - c_{31}^3 u_5. \tag{4.40}$$

Proof. By (4.31) and (4.23)

$$[\vec{h}, \widetilde{\Phi}_2] = \frac{1}{2}[\vec{h}, \vec{u}_3 + u_4\partial_{u_1} + u_5\partial_{u_2}] = \frac{1}{2}([\vec{h}, \vec{u}_3] + u_4[\vec{h}, \partial_{u_1}] + u_5[\vec{h}, \partial_{u_2}]) \bmod\Big(\text{span}(\partial_{u_1}, \partial_{u_2})\Big) =$$

$$\frac{1}{2}\Big([\vec{h}, \vec{u}_3] + \frac{1}{6}(u_4\alpha_1 + u_5\alpha_2)\epsilon_1\Big) \quad \bmod\Big(\text{span}(\vec{h}, \vec{e}, \partial_{u_1}, \partial_{u_2}, \partial_{u_3})\Big) \tag{4.41}$$

Further, from definition of adapted frame (see (2.22)) it follows that

$$[\vec{h}, \vec{u}_3] = [u_4\vec{u}_2 - u_5\vec{u}_1, \vec{u}_3] = (u_4\vec{u}_5 - u_5\vec{u}_4) +$$

$$\sum_{i=1}^{3}(c_{32}^i u_4 - c_{31}^i u_5)\vec{u}_i - \{u_3, u_4\}\vec{u}_2 + \{u_3, u_5\}\vec{u}_1$$

Using (4.31),(4.31) and (4.27), one can obtain from the last equation that

$$[\vec{h},\vec{u}_3] = -6\widetilde{\Phi}_1 + 2\big(b_1+3b\big)\widetilde{\Phi}_2 - \frac{1}{2}\Pi(b\epsilon_1+\epsilon_2) \mod\Big(\mathrm{span}(\vec{h},\vec{e},\partial_{u_1},\partial_{u_2},\partial_{u_3})\Big), \tag{4.42}$$

where Π and b_1 are as in (4.39) and (4.40) respectively. Finally, substituting (4.42) to (4.41), we get

$$[\vec{h},\widetilde{\Phi}_2] = \Big(-\frac{1}{4}\Pi b + \frac{1}{12}(\alpha_1 u_4+\alpha_2 u_5)\Big)\epsilon_1 - \frac{1}{4}\Pi\epsilon_2 + 3\widetilde{\Phi}_1 + (b_1+3b)\widetilde{\Phi}_2 \mod\Big(\mathrm{span}(\vec{h},\vec{e})\Big), \tag{4.43}$$

which concludes the proof of the lemma. □

Lemma 4.3 *The following relations hold*

$$a_{21} = -\Big(b\,b_1 + \vec{h}(b) - \frac{1}{3}a_3\Big), \tag{4.44}$$

$$a_{31} = \frac{1}{36}(\Omega-\Theta) + \frac{1}{6}(\alpha_1 u_4+\alpha_2 u_5)b - \frac{1}{4}\Pi b^2, \tag{4.45}$$

where

$$\Theta = X_5(\alpha_4)u_4^2 + \Big(X_5(\alpha_5) - X_4(\alpha_4)\Big)u_4u_5 - X_4(\alpha_5)u_5^2, \tag{4.46}$$

$$\Omega = \sum_{i=1}^{3}\Big(u_5\{u_i,u_4\} - u_4\{u_i,u_5\}\Big)\alpha_i \tag{4.47}$$

Proof. By definition of $\widetilde{\Phi}_1$ (see (4.32)

$$[\vec{h},\widetilde{\Phi}_1] = \frac{1}{6}[\vec{h},Y_4] + \vec{h}(b)\widetilde{\Phi}_2 + b[\vec{h},\widetilde{\Phi}_2]. \tag{4.48}$$

First let us calculate $[\vec{h},Y_4]$:

$$\begin{aligned}[\vec{h},Y_4] = &\left[u_4\vec{u}_2 - u_5\vec{u}_1, u_5\vec{u}_4 - u_4\vec{u}_5 + \sum_{i=1}^{3}\Big(u_5\{u_i,u_4\} - u_4\{u_i,u_5\}\Big)\partial_{u_i}\right] = \\ &-\Big(\underbrace{u_4^2[\vec{u}_2,\vec{u}_5] - u_4u_5([\vec{u}_2,u_4] + [\vec{u}_1,\vec{u}_5]) + u_5^2[\vec{u}_1,\vec{u}_4]}_{I}\Big) + \\ &\underbrace{\vec{h}(u_5)\vec{u}_4 - \vec{h}(u_4)\vec{u}_5}_{II} + \underbrace{\sum_{i=1}^{3}\Big(u_5\{u_i,u_4\} - u_4\{u_i,u_5\}\Big)[\vec{h},\partial_{u_i}]}_{III} \mod \vec{h}\end{aligned} \tag{4.49}$$

Note that the vector fields $[\vec{u}_i,\vec{u}_j]$ calculated at the points of $(D^2)^\perp$ satisfy

$$\begin{aligned}[\vec{u}_i,\vec{u}_j] = &\sum_{k=1}^{5}\Big(c_{ji}^k\vec{u}_k - X_k(c_{ji}^4u_4 + c_{ji}^5u_5)\partial_{u_k}\Big) = \sum_{k=1}^{5}c_{ji}^k\vec{u}_k + \frac{1}{6}\Big(X_5(c_{ji}^4)u_4^2 + \\ &\big(X_5(c_{ji}^5) - X_4(c_{ji}^4)\big)u_4u_5 - X_4(c_{ji}^5)u_5^2\Big)\epsilon_1 \mod\Big(\mathrm{span}(\partial_{u_1},\partial_{u_2},\partial_{u_3}),\vec{e}\Big)\end{aligned} \tag{4.50}$$

(the last equality here was obtained with the help of (4.19)). Substituting the last formula in the term I of (4.49) one can get by direct calculation that

$$I=\sum_{k=1}^{5}\alpha_k\vec u_k+\frac16\Theta\epsilon_1\mod\Big(\mathrm{span}(\partial_{u_1},\partial_{u_2},\partial_{u_3}),\vec e\Big),\tag{4.51}$$

where α_k are exactly as in (4.23) and Θ is defined by (4.46).

Using (4.31) and (4.27), one can obtain from (4.51) that

$$\begin{gathered}-I+II=(\vec h(u_5)-\alpha_4)\vec u_4-(\vec h(u_4)+\alpha_5)\vec u_5-2\alpha_3\widetilde\Phi_2+\\ \frac12(\alpha_1u_4+\alpha_2u_5)(b\epsilon_1+\epsilon_2)-\frac16\Theta\epsilon_1\mod\Big(\mathrm{span}(\partial_{u_1},\partial_{u_2},\partial_{u_3},\vec h,\vec e)\Big)\end{gathered}\tag{4.52}$$

Then from relations (4.29),(4.30), and (4.32) it follows easily that

$$\begin{gathered}-I+II=\left(-\frac16\Theta+\frac12(\alpha_1u_4+\alpha_2u_5)b\right)\epsilon_1+\frac12(\alpha_1u_4+\alpha_2u_5)\epsilon_2+\\ 18b\widetilde\Phi_1-(2a_3+18b^2)\widetilde\Phi_2\mod\Big(\mathrm{span}(\partial_{u_1},\partial_{u_2},\partial_{u_3},\vec h,\vec e)\Big)\end{gathered}\tag{4.53}$$

Note also that by (4.20) and (4.47) the term III of (4.49) satisfies

$$III=\frac16\Omega\epsilon_1\mod\Big(\mathrm{span}(\partial_{u_1},\partial_{u_2},\partial_{u_3},\vec h,\vec e)\Big)\tag{4.54}$$

Substituting (4.53) and (4.54) in (4.49) and then in (4.48) and using (4.43) one can obtain without difficulties that

$$\begin{gathered}[\vec h,\widetilde\Phi_1]=\left(\frac1{36}(\Omega-\Theta)+\frac16(\alpha_1u_4+\alpha_2u_5)b-\frac14\Pi b^2\right)\epsilon_1+\\ \left(-\frac14\Pi b+\frac1{12}(\alpha_1u_4+\alpha_2u_5)\right)\epsilon_2+\Big(b\,b_1+\vec h(b)-\frac13a_3\Big)\widetilde\Phi_2,\end{gathered}\tag{4.55}$$

which concludes the proof of the lemma. □

Substituting relations (4.36), (4.37), and (4.44) in (4.4) it is not difficult to obtain that the Ricci curvature $\rho(\lambda)$ of the reduced Jacobi curve $J_\gamma(e^{t\vec h}\lambda)$ at $t=0$ satisfies

$$\rho=-\frac4{15}(a_3-\frac12\Pi-\frac12\vec h(b_1)-\frac92\vec h(b)+\frac12b_1^2+\frac92b^2).\tag{4.56}$$

Finally substituting the last formula and formulas (4.36), (4.38), (4.44), and (4.45) in (4.5) we obtain the following expression for the density A of the fundamental form:

$$\begin{aligned}\frac{35}{36}A\;=\;&\frac1{36}(\Theta-\Omega)-\frac16(\alpha_1u_4+\alpha_2u_5)b+\frac14\Pi b^2+\\ &\frac1{100}\Big(a_3-\frac12\Pi-\frac12\vec h(b_1)-\frac92\vec h(b)+\frac12b_1^2+\frac92b^2\Big)^2-\\ &\frac1{60}\vec h\circ\vec h\Big(a_3-\frac12\Pi-\frac12\vec h(b_1)-\frac92\vec h(b)+\frac12b_1^2+\frac92b^2\Big)-\\ &\frac14\Big(b\,b_1+\vec h(b)-\frac13a_3\Big)^2+\frac1{36}\vec h(\alpha_1u_4+\alpha_2u_5)-\frac1{12}\vec h(\Pi b)+\\ &\frac1{12}\vec h\circ\vec h\Big(b\,b_1+\vec h(b)-\frac13a_3\Big)+\frac1{12}\vec h\Big(\big(b\,b_1+\vec h(b)-\frac13a_3\big)(b_1+3b)\Big).\end{aligned}\tag{4.57}$$

Remark 4.1 Analyzing all functions involved in (4.57), it is not hard to see that the coefficients of fundamental form (which is polynomial of degree 4 in u_4 and u_5) can be expressed by the following structural functions of the chosen adapted frame:

$$\begin{aligned}&c_{3j_1}^{k_1}, \quad j_1 = 1, 2;\ k_1 = 1, 2, 3;\\ &c_{i_2j_2}^{k_2}, \quad i_2 = 4, 5;\ j_2 = 1, 2;\ k_2 = 1 \dots 5;\\ &c_{i_33}^{k_3}, \quad i_3 = 4, 5;\ k_3 = 4, 5.\end{aligned} \tag{4.58}$$

Note also that the structural functions of the third raw of (4.58) can be expressed by structural functions of the second row. □

Remark 4.2 If the chosen frame is strongly adapted to the distribution, then $c_{3j}^k = 0$ for $j = 1, 2$ and therefore

$$b_1 = 0$$
$$\Pi = -u_5\{u_3, u_4\} + u_4\{u_3, u_5\}.$$

□

5 Proof of Theorem 2

Fix some coframe $\{\omega_i\}_{i=1}^5$ satisfying (3.2). More precisely, take some section of the bundle $\mathcal{K}_1(M)$ over M. We will think of all 1-forms under consideration and of the equations (3.2),(3.6)-(3.8) as the objects restricted on this section. Let $\{\widetilde{X}_k\}_{k=1}^5$, be the frame on M dual to the coframe $\{\omega_i\}_{i=1}^5$, namely,

$$\omega_i(\widetilde{X}_k) = \delta_{i,k}. \tag{5.1}$$

Let

$$X_k = \widetilde{X}_{5-k+1}, \quad 1 \le k \le 5 \tag{5.2}$$

Any frame $\{X_i\}_{i=1}^5$ obtained in such way will be called *Cartan's frame* of the distribution D. Note that

$$D(q) = span(\widetilde{X}_4(q), \widetilde{X}_5(q)) = span(X_1(q), X_2(q)), \quad q \in M. \tag{5.3}$$

Using (5.1) and the well-known formula

$$d\omega(V_1, V_2) = V_1\omega(V_2) - V_2\omega(V_1) - \omega([V_1, V_2]) \tag{5.4}$$

one can find from (3.2) the commutative relations for the frame $\{Y_k\}_{k=1}^5$ and therefore for the frame $\{X_k\}_{k=1}^5$. In particular,

$$[X_1, X_2] = X_3 + \Big(\overline{\omega}_3(X_1) - \overline{\omega}_4(X_2)\Big)X_1 + \Big(\overline{\omega}_1(X_1) - \overline{\omega}(X_2)\Big)X_2 \tag{5.5}$$

$$[X_1, X_3] = X_4 - \left(\frac{4}{3}\overline{\omega}_5(X_1) + \overline{\omega}_4(X_3)\right)X_1 + \left(\frac{4}{3}\overline{\omega}_6(X_1) - \overline{\omega}_2(X_3)\right)X_2 + \Big(\overline{\omega}_1 + \overline{\omega}_4\Big)(X_1)X_3 \tag{5.6}$$

$$[X_2, X_3] = X_5 - \left(\frac{4}{3}\overline{\omega}_5(X_2) + \overline{\omega}_3(X_3)\right)X_1 + \left(\frac{4}{3}\overline{\omega}_6(X_2) - \overline{\omega}_1(X_3)\right)X_2 + \Big(\overline{\omega}_1 + \overline{\omega}_4\Big)(X_2)X_3 \tag{5.7}$$

These formulas together with (5.3) imply that any Cartan's frame $\{X_i\}_{i=1}^5$ is adapted to the distribution D. So, starting with some Cartan's frame $\{X_i\}_{i=1}^5$, one can apply the formula (4.57) for computation of the fundamental form. From (3.2) the structural functions of the frame $\{X_i\}_{i=1}^5$ can be expressed in terms of the forms $\overline{\omega}_j$ and the vector fields X_i.

Hence our fundamental form can be expressed in terms of forms $\overline{\omega}_j$ and vector fields X_k. We will compare this expression with expression of Cartan's tensor $\mathcal{F}$ which can be obtained by substitution of some of X_k's into the formulas (3.6)-(3.8).

By Remark 4.2 in order to calculate the fundamental form we will need the following commutative relations in addition to (5.5)-(5.7):

$$\begin{aligned}[X_1, X_4] \quad &= \quad \Big(\overline{\omega}_7(X_1) - \overline{\omega}_4(X_4)\Big)X_1 - \overline{\omega}_2(X_4)X_2 + \overline{\omega}_6(X_1)X_3 + \\ &\quad \Big(\overline{\omega}_1 + 2\overline{\omega}_4\Big)(X_1)X_4 + \overline{\omega}_2(X_1)X_5,\end{aligned} \tag{5.8}$$

$$\begin{aligned}[X_1, X_5] \quad &= \quad -\overline{\omega}_4(X_5)X_1 + \Big(\overline{\omega}_7(X_1) - \overline{\omega}_2(X_5)\Big)X_2 + \overline{\omega}_5(X_1)X_3 + \\ &\quad \overline{\omega}_3(X_1)X_4 + \Big(2\overline{\omega}_1 + \overline{\omega}_4\Big)(X_1)X_5,\end{aligned} \tag{5.9}$$

$$\begin{aligned}[X_2, X_4] \quad &= \quad \Big(\overline{\omega}_7(X_2) - \overline{\omega}_3(X_4)\Big)X_1 - \overline{\omega}_1(X_4)X_2 + \overline{\omega}_6(X_2)X_3 + \\ &\quad \Big(\overline{\omega}_1 + 2\overline{\omega}_4\Big)(X_2)X_4 + \overline{\omega}_2(X_2)X_5,\end{aligned} \tag{5.10}$$

$$\begin{aligned}[X_2, X_5] \quad &= \quad -\overline{\omega}_3(X_5)X_1 + \Big(\overline{\omega}_7(X_2) - \overline{\omega}_1(X_5)\Big)X_2 + \overline{\omega}_5(X_2)X_3 + \\ &\quad \overline{\omega}_3(X_2)X_4 + \Big(2\overline{\omega}_1 + \overline{\omega}_4\Big)(X_2)X_5,\end{aligned} \tag{5.11}$$

$$[X_3, X_4] = (\overline{\omega}_1 + 2\overline{\omega}_4)(X_3)X_4 + \overline{\omega}_2(X_3)X_5 \mod\Big(\mathrm{span}(X_1, X_2, X_3)\Big), \tag{5.12}$$

$$[X_3, X_5] = \overline{\omega}_3(X_3)X_4 + (2\overline{\omega}_1 + \overline{\omega}_4)(X_3)X_5, \mod\Big(\mathrm{span}(X_1, X_2, X_3)\Big). \tag{5.13}$$

These relations can be easily obtained by applying formulas (5.1) and (5.4) to (3.2).

First note that for Cartan's frame from commutative relations (5.8)-(5.11) it follows that

$$b = (\overline{\omega}_1 + \overline{\omega}_4)(X_2)u_4 - (\overline{\omega}_1 + \overline{\omega}_4)(X_1)u_5. \tag{5.14}$$

Further, from (4.40), (5.6), (5.7), and (5.14) it follows that for Cartan's frames

$$b_1 = b \tag{5.15}$$

In addition, substituting (5.6)-(5.11) in (4.23) and (4.39) one can obtain by direct computations that for Cartan's frames

$$\Pi = -\frac{4}{3}\alpha_3 \tag{5.16}$$

After substitution of (5.15) and (5.16) in (4.57) most of the terms of (4.57) are surprisingly canceled and we obtain that in Cartan's frame the density A of fundamental form satisfies

$$A = \frac{1}{35}\Big(\Theta + \vec{h}(\alpha_1 u_4 + \alpha_2 u_5) - \Omega - 6(\alpha_1 u_4 + \alpha_2 u_5)b\Big) \tag{5.17}$$

Let us try to express the density A in the most convenient way in terms of forms $\overline{\omega}_j$ and vector fields X_k. For this first note that the term $\vec{h}(\alpha_1 u_4 + \alpha_2 u_5)$ can be written in the following form:

$$\begin{aligned}\vec{h}(\alpha_1 u_4 + \alpha_2 u_5) \quad &= \quad \Theta_1 + \vec{h}(u_4)(\alpha_1 + u_4 \frac{\partial}{\partial u_4}\alpha_1 + u_5 \frac{\partial}{\partial u_4}\alpha_2) + \\ &\quad\quad \vec{h}(u_5)(\alpha_2 + u_4 \frac{\partial}{\partial u_5}\alpha_1 + u_5 \frac{\partial}{\partial u_5}\alpha_2),\end{aligned} \tag{5.18}$$

where

$$\Theta_1 = X_2(\alpha_1)u_4^2 + \Big(X_2(\alpha_2) - X_1(\alpha_1)\Big)u_4 u_5 - X_1(\alpha_2)u_5^2. \tag{5.19}$$

Note that Θ and Θ_1 are the only terms of $35A$ containing expression of the form $X_k\overline{\omega}_i(X_j)$. Let us analyze the sum $\Theta + \Theta_1$. For this let us introduce some notations. Given 1-form ω and two vector fields denote by $\mathcal{S}(\omega, V_1, V_2)$ the following expression

$$\mathcal{S}(\omega, V_1, V_2) \stackrel{def}{=} V_1\omega(V_2) - V_2\omega(V_1) \tag{5.20}$$

Denote also by $\mathcal{W}_{u_4,u_5}$ the following 1-form on M:

$$\mathcal{W}_{u_4,u_5} = u_4^2\overline{\omega}_3 + u_4 u_5(\overline{\omega}_1 - \overline{\omega}_4) - u_5^2\overline{\omega}_2 \tag{5.21}$$

Then from (4.23), (4.46),(5.19), and commutative relations (5.8)-(5.11) one can obtain by direct calculations that

$$\begin{aligned}\Theta + \Theta_1 \quad &= \quad \mathcal{S}(\mathcal{W}_{u_4,u_5}, X_5, X_2)u_4^2 + \Big(\mathcal{S}(\mathcal{W}_{u_4,u_5}, X_1, X_5) + \\ &\quad\quad \mathcal{S}(\mathcal{W}_{u_4,u_5}, X_2, X_4)\Big)u_4 u_5 + \mathcal{S}(\mathcal{W}_{u_4,u_5}, X_4, X_1)u_5^2\end{aligned} \tag{5.22}$$

The "commutative" nature of $\Theta + \Theta_1$ suggests an idea to compare the polynomial $35A$ with the following polynomial of degree 4 in u_4 and u_5:

$$\begin{aligned}\mathcal{B} \quad &\stackrel{def}{=} \quad d\mathcal{W}_{u_4,u_5}(X_5, X_2)u_4^2 + \Big(d\mathcal{W}_{u_4,u_5}(X_1, X_5) + \\ &\quad\quad d\mathcal{W}_{u_4,u_5}(X_2, X_4)\Big)u_4 u_5 + d\mathcal{W}_{u_4,u_5}(X_4, X_1)u_5^2\end{aligned} \tag{5.23}$$

(the polynomial $\mathcal{B}$ is obtained from the righthand side of (5.22) by replacing the operation $\mathcal{S}$ with the operation of exterior differential d). Substituting expressions (4.16) for $\vec{h}$, (4.23) for α_i, (5.14) for b, formulas (5.18) and (5.22) in (5.17), then formulas (5.21) and identity (5.4) in (5.23) and using commutative relations (5.8)-(5.11), one can get by long but direct calculations the following

$$\begin{aligned}35A - \mathcal{B} \quad &= \quad \Xi_{u_4,u_5}(X_5, X_2)u_4^2 + \Big(\Xi_{u_4,u_5}(X_1, X_5) + \\ &\quad\quad \Xi_{u_4,u_5}(X_2, X_4)\Big)u_4 u_5 + \Xi_{u_4,u_5}(X_4, X_1)u_5^2,\end{aligned} \tag{5.24}$$

where Ξ_{u_4,u_5} is the following 2-form on M:

$$\Xi_{u_4,u_5} = u_4^2\varpi_3 \wedge (\varpi_1 - \varpi_4) - 2u_4u_5\varpi_3 \wedge \varpi_2 + u_5^2\varpi_2 \wedge (\varpi_1 - \varpi_4). \tag{5.25}$$

On the other hand, from relations (3.6)-(3.8) and duality (see relations (5.1) and (5.2)) it follows without difficulties that

$$\begin{aligned} &\mathcal{B} + \Xi_{u_4,u_5}(X_5, X_2)u_4^2 + \Big(\Xi_{u_4,u_5}(X_1, X_5) + \Xi_{u_4,u_5}(X_2, X_4)\Big)u_4u_5 + \\ &\Xi_{u_4,u_5}(X_4, X_1)u_5^2 = -\Big(A_1u_4^4 - 4A_2u_4^3u_5 + 6A_3u_4^2u_5^2 - 4A_4u_4u_5^3 + A_5u_5^4\Big), \end{aligned} \tag{5.26}$$

where coefficients A_i are exactly as in (3.9). So,

$$35A = -\Big(A_1u_4^4 - 4A_2u_4^3u_5 + 6A_3u_4^2u_5^2 - 4A_4u_4u_5^3 + A_5u_5^4\Big) \tag{5.27}$$

Finally, we can find the tangential fundamental form $\mathring{A}_q$. For given $v \in D(q)$ by duality one has $v = \omega_4(v)X_2 + \omega_5(v)X_1$. Take $\lambda \in (D^2)^\perp(q)$ with

$$u_4 = \omega_4(v), \quad u_5 = -\omega_5(v). \tag{5.28}$$

Let $\vec{h} = \vec{h}_{X_1,X_2}$ as in (2.27). Then by construction $\pi_*\vec{h}(\lambda) = v$. Further from (2.29) and (2.28)

$$35\mathring{A}_q(v) = 35\mathcal{A}_\lambda(\vec{h}(\lambda)) = 35A(\lambda)$$

In order to calculate $A(\lambda)$ one substitutes (5.28) in (5.27). Comparing the result of this substitution with (3.9), one obtains (3.10), which concludes the proof of Theorem 2.

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