# Parametrized curves in Lagrange Grassmannians

Igor Zelenko, Chengbo Li
*S.I.S.S.A., Via Beirut 2-4, 34014, Trieste, Italy*

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**Abstract**

Curves in Lagrange Grassmannians naturally appear when one studies Jacobi equations for extremals, associated with geometric structures on manifolds. We fix integers $d_i$ and consider curves $\Lambda(t)$ for which at each $t$ the derivatives of order $\leq i$ of all curves of vectors $\ell(t) \in \Lambda(t)$ span a subspace of dimension $d_i$. We will describe the construction of a complete system of symplectic invariants for such parametrized curves, satisfying a certain genericity assumption, and give applications to geometric structures, including sub-Riemannian and sub-Finslerian structures.

**Résumé**

**Courbes paramétrées dans les Grassmanniennes Lagrangiennes**
Les courbes dans les Grassmanniennes Lagrangiennes apparaissent naturellement lors de l'étude intrinsèque des "équations de Jacobi pour les extremas", associées à des structures géométriques sur les variétés différentielles. Nous fixons des entiers $d_i$ et considérons les courbes $\Lambda(t)$ pour lesquelles en chaque $t$ les dérivées d'ordre $\leq i$ des $\ell(t) \in \Lambda(t)$ engendrent un sous-espace de dimension $d_i$. Nous décrirons la construction d'un système complet d'invariants symplectiques pour de telles courbes paramétrées vérifiant une condition de généricité, et nous donnerons des applications à la géométrie différentielle de structures géométriques, incluant les structures sous-riemanniennes et sous-finsleriennes.

## Version française abrégée

Soit $\Lambda(t)$, $t \in [0, T]$, une courbe dans la Grassmannienne Lagrangienne $L(W)$ de l'espace vectoriel symplectique $(W, \omega)$. En étudiant les courbes dans les Grassmanniennes Lagrangiennes , on peut développer de façon unifiée ([1]) la géométrie différentielle de structures géométriques sur les variétés. Notons par $\mathfrak{S}(\Lambda)$ l'ensemble des courbes $\ell(t)$ dans $W$ telles que $\ell(t) \in \Lambda(t)$ pour tout $t$. Soit $\Lambda^{(j)}(\tau) = \mathrm{span}\Big\{\frac{d^i}{d\tau^i}\ell(\tau)| : \ell \in \mathfrak{S}(\Lambda),\, 0 \leq i \leq j\Big\}$. On dit qu'une courbe $\Lambda(t)$ est à *croissance constante*, si les dimensions des $\Lambda^{(j)}(t)$ sont indépendentes de $t$. L'objectif de cet article est de décrire un système complet d'invariants d'une courbe $\Lambda(t)$ à croissance constante, satisfaisant l'hypothèse

*Email addresses:* zelenko@sissa.it (Igor Zelenko), chengbo@sissa.it (Chengbo Li).

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de généricité (G), pour l'action du groupe symplectique de $W$. Soit $\Lambda^{(-j)}(t) = \big(\Lambda^{(j)}(t)\big)^{\angle}$, où $\big(\Lambda^{(j)}(t)\big)^{\angle}$ dénote l'orthogonal symplectique de $\Lambda^{(j)}(t)$. Soit $\mathrm{Gr}_j(t) = \Lambda^{(j)}(t)/\Lambda^{(j-1)}(t)$. L'espace gradué $\mathrm{Gr}\,W(t)$ somme des $\mathrm{Gr}_j(t)$ est naturellement muni d'une structure symplectique et d'un endomorphisme de degré un $\delta$, appartenant à l'algèbre de Lie du groupe symplectique. Pour une courbe satisfaisant la condition (G), il existe un tuple d'entiers $\{r_i^+, r_i^-\}_{i=1}^k$ tel que le groupe $G_{\mathrm{Aut}}$ des transformations symplectiques de $\mathrm{Gr}\,W(t)$, préservant la graduation et commutant avec $\delta$, est isomorphe à $O(r_1^+, r_1^-) \times \ldots \times O(r_k^+, r_k^-)$. Pour chaque courbe nous construisons une connexion canonique sur le fibré vectoriel sur $[0,T]$ de fibres les $\mathrm{Gr}\,W(t)$ et une décomposition symplectique canonique de la filtration $\{\Lambda^{(j)}(t)\}_{j\in\mathbb{Z}}$ de $W$, i.e. un isomorphisme symplectique canonique $A_t : \mathrm{Gr}\,W(t) \mapsto W$ compatible avec la filtration et tel que l'application graduée induite est l'identité. Pour le fibré principal constant sur $[0,T]$ des repères symplectiques de $W$, ceci fournit une réduction du groupe structural à $G_{\mathrm{Aut}}$ et une connexion sur le $G_{\mathrm{Aut}}$ fibré principal correspondant (un $G_{\mathrm{Aut}}-$fibré de repères mobiles symplectiques). Les invariants qui apparaissent dans l'équation structurale de ces repères mobiles constituent un système complet d'invariants symplectiques de la courbe.

## 1. Introduction

Let $(W,\omega)$ be a real symplectic vector space. The Lagrange Grassmannian $L(W)$ of $W$ is the space of Lagrangian subspaces of $W$. The symplectic group $Sp(W)$ acts on $L(W)$. Let $\Lambda : [0,T] \mapsto L(W)$ be a parametrized smooth curve in $L(W)$. Denote by $\mathfrak{S}(\Lambda)$ the set of all smooth curves $\ell(t)$ in $W$ such that $\ell(t) \in \Lambda(t)$ for all $t$. Let $\Lambda^{(i)}(\tau) = \mathrm{span}\Big\{\frac{d^j}{d\tau^j}\ell(\tau) \mid : \ell \in \mathfrak{S}(\Lambda),\, 0 \le j \le i\Big\}$. An analogous construction can be done for smooth curves in any Grassmannian. By construction $\Lambda^{(i-1)}(\tau) \subseteq \Lambda^{(i)}(\tau)$. We will consider only curves $\Lambda(t)$ of *constant growth*, i.e. such that the dimensions of the subspaces $\Lambda^{(i)}(t)$ are independent of $t$. This assumption always holds for $t$ in some non empty open subset of $[0,T]$. Constant growth implies that $\dim \Lambda^{(i+1)}(t) - \dim \Lambda^{(i)}(t) \le \dim \Lambda^{(i)}(t) - \dim \Lambda^{(i-1)}(t)$ and we define as follows the Young diagram $D$ of $\Lambda(t)$: the number of boxes in the $i$th column of $D$ is equal to $\dim \Lambda^{(i)} - \dim \Lambda^{(i-1)}$. Our main problem is *to find a complete system of invariants w.r.t. the natural action of $Sp(W)$ on $L(W)$ of a parametrized curve $\Lambda(t)$ in $L(W)$ with given Young diagram $D$*. This problem is a particular case of the problem of finding invariants of curves in homogeneous spaces. A general procedure for the latter problem was developed already by E. Cartan with his method of moving frames. By studying curves in Lagrange Grasmannians, one can develop in the unified way the differential geometry of geometric structures on manifolds ([1]). Here by a geometric structure on a manifold $M$ we mean a submanifold $\mathcal{V}$ of the tangent bundle $TM$ transversal to the fibers. Examples are sub-Riemannian (respectively sub-Finslerian) structures: the case where for any $q \in M$ the set $\mathcal{V} \cap T_qM$ is an ellipsoid centered at the origin (respectively the boundary of a convex body) in a linear subspace $\mathcal{D}_q$ in $T_qM$. The problem stated above was previously solved only in the following two cases: the case when the Young diagram $D$ consists of one column, which corresponds to curves appearing in the case of Riemannian or Finslerian structures ([1]), and the case when the Young diagram consists of one row ([2]). We solve the problem for parametrized curves in $L(W)$ with arbitrary Young diagram, satisfying the genericity condition (G) below. This condition holds automatically for the so-called monotonic curves, i.e., when the velocities of a curve are semidefinite quadratic forms at any point (under the identification of the tangent space to $L(W)$ at $\Lambda$ with the space of quadratic forms on $\Lambda$). This is the case for curves corresponding to sub-Riemannian or sub-Finslerian structures. Our use of Young diagrams is not related to representations of the symmetric group. Rather, they are convenient to describe normalization conditions for moving frames associated with a curve.

## 2. The main results

Let $\Lambda(t)$ be a smooth curve in the Grassmannian $G_k(W)$ of all $k$-dimensional subspaces in $W$. Given a subspace $V$ in $W$ denote by $V^\perp$ the annihilator of $V$ in the dual space $W^*$: $V^\perp = \{p \in W^* : \langle p, v\rangle = 0, \forall\, v \in V\}$. Set $\Lambda^{(-j)}(t) = \big((\Lambda(t)^\perp)^{(j)}\big)^\perp$ for $j \ge 0$. We get the nondecreasing filtration $\{\Lambda^{(j)}(t)\}_{j\in\mathbb{Z}}$ of $W$. Recall that the velocity

$\frac{d}{dt}\Lambda(t)$ can be identified with an element of $\mathrm{Hom}\big(\Lambda(t), W/\Lambda(t)\big)$ in the following way: $\frac{d}{dt}\Lambda(t)v = [\ell'(0)]$, where $\ell \in \mathfrak{S}(\Lambda)$, $\ell(0) = v$, and $[\ell'(0)]$ is the image of $\ell'(0)$ in $W/\Lambda(t)$. In the case of a curve $\Lambda(t)$ in the Langrange Grassmannian $L(W)$ the velocity $\frac{d}{dt}\Lambda(t)$ can be identified with a quadratic form on $\Lambda(t)$ by $\frac{d}{dt}\Lambda(t)v = \omega(\ell'(0), v)$, where $\ell$ is as above. If the curve $\Lambda(t)$ has constant growth, then the subspaces $\Lambda^{(-j)}(t)$, $j > 0$ can be also defined recursively by $\Lambda^{(-j)}(t) = \mathrm{Ker}\frac{d}{dt}\Lambda^{(-j+1)}(t)$. Moreover, if we define $\mathrm{Gr}_j(t) := \Lambda^{(j)}(t)/\Lambda^{(j-1)}(t)$, then the velocity $\frac{d}{dt}\Lambda^{(j)}(t)$ factors through a map $\delta$ from $\mathrm{Gr}_j(t)$ to $\mathrm{Gr}_{j+1}(t)$. This map is surjective for $j \geq 0$ and injective for $j \leq 0$. If $\Lambda(t)$ is a curve in $L(W)$, then we have $\Lambda^{(-j)}(t) = \big(\Lambda^{(j)}(t)\big)^{\angle}$, where $\big(\Lambda^{(j)}(t)\big)^{\angle}$ denotes the symplectic orthogonal of $\Lambda^{(j)}(t)$ (w.r.t the symplectic form $\omega$). In this case the form $\omega$ induces a symplectic form $\bar{\omega}$ on the graded space $\mathrm{Gr}\, W(t) = \bigoplus_{j\in\mathbb{Z}} \mathrm{Gr}_j(t)$ in the following way: if $\bar{x} \in \mathrm{Gr}_j(t)$ and $\bar{y} \in \mathrm{Gr}_{\tilde{j}}(t)$ with $j + \tilde{j} = 1$, then $\bar{\omega}(\bar{x}, \bar{y}) := w(x, y)$, where $x$ and $y$ are representatives of $\bar{x}$ and $\bar{y}$ in $\Lambda^{(j)}(t)$ and $\Lambda^{(\tilde{j})}(t)$ respectively; if $j + \tilde{j} \neq 1$, then $\bar{\omega}(\bar{x}, \bar{y}) = 0$. In general, let $\bigoplus_{j\in\mathbb{Z}} X_j$ be a graded symplectic space and $N$ in its symplectic Lie algebra be such that $N(X_j) \subseteq X_{j+1}$. The pair $\big(\bigoplus_{j\in\mathbb{Z}} X_j, N\big)$ is called a *symbol*. The symbol $\big(\bigoplus_{j\in\mathbb{Z}} \mathrm{Gr}_j(t), \delta\big)$ is said to be the *symbol of the curve* $\Lambda(t)$ *at the point* $t$.

We do not lose much by assuming that there exists an integer $p$ such that $\Lambda^{(p)}(t) = W$. Otherwise, if $\Lambda^{(p+1)}(t) = \Lambda^{(p)}(t) \subset W$, then the subspace $V = \Lambda^{(p)}(t)$ does not depend on $t$ and one can work with the curve $\Lambda(t)/V^{\angle}$ in the symplectic space $V/V^{\angle}$ instead of $\Lambda(t)$. We will say that the curve $\Lambda(t)$ in $L(W)$ satisfies *condition (G)* if it has constant growth, is such that $\Lambda^{(p)}(t) = W$ for some $p$, and if further for any $j \geq 0$ the map $\delta^{2j+1} : \mathrm{Gr}_{-j}(t) \mapsto \mathrm{Gr}_{j+1}(t)$ is an isomorphism. As the pairing between $\mathrm{Gr}_{-j}(t)$ and $\mathrm{Gr}_{j+1}(t)$ is nondegenerate and the form $\bar{\omega}(\delta^{2j+1}\bar{x}, \bar{y})$ on $\mathrm{Gr}_{-j}(t)$ is symmetric, this amounts to require that for each $j \geq 0$ the quadratic form $\bar{x} \mapsto \bar{\omega}(\delta^{2j+1}\bar{x}, \bar{x})$ on $\mathrm{Gr}_{-j}(t)$ is nondegenerate or, equivalently, the quadratic form $\bar{x} \mapsto \bar{\omega}(\delta\bar{x}, \bar{x})$, defined on the subspace $\delta^j\big(\mathrm{Gr}_{-j}(t)\big)$ of $\mathrm{Gr}_0(t)$, is nondegenerate (the latter equivalence follows from the identity $\bar{\omega}(\delta^{2j+1-s}\bar{x}, \delta^s\bar{x}) = (-1)^s\bar{\omega}(\delta^{2j+1}\bar{x}, \bar{x})$ for all $\bar{x} \in \mathrm{Gr}_{-j}(t)$). As a consequence, monotonic curves $\Lambda(t)$ of constant growth with $\Lambda^{(p)}(t) = W$ for some $p$ satisfy condition (G). Indeed, if the velocity $\frac{d}{dt}\Lambda(t)$ is semidefinite, then the quadratic form $\bar{x} \mapsto \bar{\omega}(\delta\bar{x}, \bar{x})$ on $\mathrm{Gr}_0(t)$ is definite and therefore its restriction to any subspace of $\mathrm{Gr}_0(t)$ is definite. Germs of curves at a point $\tau$ satisfying condition (G) are generic among all germs of curves at $\tau$ with given Young diagram $D$ such that the number of boxes of $D$ is equal to $\frac{1}{2}\dim W$.

From now on $\Lambda : [0, T] \mapsto L(W)$ will be a curve having the Young diagram $D$ and satisfying condition (G). Let the length of the rows of $D$ be $p_1$ repeated $r_1$ times, $p_2$ repeated $r_2$ times, …, $p_k$ repeated $r_k$ times with $p_1 > p_2 > \ldots > p_k$. *The reduction of the Young diagram $D$* is the Young diagram $\Delta$, consisting of $k$ rows such that the $i$th row has $p_i$ boxes. Let $\sigma_i$ be the last box of the $i$th row of $\Delta$ and $c_j$ be the number of boxes in the $j$th column of $\Delta$. For any $1 \leq i \leq k$ denote by $\mathcal{P}_{\sigma_i}(t) \subset \mathrm{Gr}_{1-p_i}(t)$ the kernel of the map $\delta^{2p_i} : \mathrm{Gr}_{1-p_i}(t) \mapsto \mathrm{Gr}_{p_i+1}(t)$. One has $\dim \mathcal{P}_{\sigma_i} = r_i$ and isomorphisms

$$\bigoplus_{i=1}^{c_j} \mathcal{P}_{\sigma_i}(t) \overset{\varphi_j}{\overset{\sim}{\mapsto}} \mathrm{Gr}_{1-j}(t) \overset{\delta^{2j-1}}{\overset{\sim}{\mapsto}} \mathrm{Gr}_j(t), \tag{1}$$

where $\varphi_j(\bar{x}) \overset{def}{=} \delta^{p_i - j}(\bar{x})$ for any $\bar{x} \in \mathcal{P}_{\sigma_i}(t)$. The subspace $\mathcal{P}_{\sigma_i}(t)$ of $\mathrm{Gr}_{1-p_i}(t)$ is in fact the orthogonal complement of $\delta\big(\mathrm{Gr}_{-p_i}(t)\big)$ w.r.t. the nondegenerate symmetric bilinear form $\langle \bar{x}, \bar{y} \rangle = (-1)^{p_i - 1}\bar{\omega}(\delta^{2p_i - 1}\bar{x}, \bar{y})$. So, the restriction of this form to $\mathcal{P}_{\sigma_i}(t)$ is nondegenerate as well. This restriction defines the *canonical pseudo-Euclidean structure $Q_i$ on $\mathcal{P}_{\sigma_i}$*. The negative inertia index $r_i^-$ of $Q_i$ is said to be the *$i$th negative index* of the curve $\Lambda$. The symbols of the curve $\Lambda$ at different points are all isomorphic. Their isomorphism class is determined by the Young diagram $D$ and the tuple of negative indexes $\{r_i^-\}_{i=1}^k$ of the curve $\Lambda$. Further, for any $1 \leq i \leq k$, let $W_i(t) = \big(\Lambda^{(1-p_1)}\big)^{(2p_1 - 1)}(t) + \ldots + \big(\Lambda^{(1-p_i)}\big)^{(2p_i - 1)}(t)$. Then for a curve satisfying condition (G) the restriction of the symplectic form $\omega$ to the subspace $W_i(t)$ is nondegenerate for any $1 \leq i \leq k$. Set $V_{\sigma_i}(t) \overset{def}{=} \Lambda^{(1-p_i)}(t) \cap W_{i-1}(t)^{\angle}$. The subspace $V_{\sigma_i}(t)$ is a lift of the subspaces $\mathcal{P}_{\sigma_i}(t)$ from $\mathrm{Gr}_{-p_i+1}(t) = \Lambda^{(1-p_i)}(t)/\Lambda^{(-p_i)}(t)$ to $\Lambda^{1-p_i}(t)$. Namely, if $\mathrm{pr}_j : \Lambda^{(j)}(t) \mapsto \mathrm{Gr}_j(t)$ denotes the canonical projection to the factor space, then $\mathrm{pr}_{1-p_i}|_{V_{\sigma_i}(t)}$ defines an isomorphism between $V_{\sigma_i}(t)$ and $\mathcal{P}_{\sigma_i}(t)$. Let $A_t : \bigoplus_{i=1}^{k} \mathcal{P}_{\sigma_i}(t) \mapsto W$ be the linear map such that $A_t|_{\mathcal{P}_{\sigma_i}(t)}$ is the inverse map of $\mathrm{pr}_{1-p_i}|_{V_{\sigma_i}(t)}$ for any $1 \leq i \leq k$. The vector bundle $\mathcal{P}_{\sigma_i}$ over $[0,T]$ with fibers $\mathcal{P}_{\sigma_i}(t)$ has a unique connection such that the corresponding parallel transform preserves the canonical pseudo-Euclidean structures on the fibers and such that for any two horizontal sections $s_1(t)$ and $s_2(t)$ one has $\omega\Big(\frac{d^{p_i}}{dt^{p_i}}A_t\big(s_1(t)\big), \frac{d^{p_i}}{dt^{p_i}}A_t\big(s_2(t)\big)\Big) = 0$. Using identifications (1), we obtain also a canonical linear connection on the vector bundle $\operatorname{Gr} W$ over $[0,T]$ with fibers $\operatorname{Gr} W(t)$.

We say that a symplectic linear map $B : \operatorname{Gr} W(t) \mapsto W$ is a *symplectic splitting of the filtration* $\{\Lambda^{(j)}(t)\}_{j\in\mathbb{Z}}$, if $B$ is compatible with the filtration $\{\Lambda^{(j)}(t)\}_{j\in\mathbb{Z}}$, i.e. $B\big(\operatorname{Gr}_j(t)\big) \subset \Lambda^{(j)}(t)$, and satisfies $\operatorname{pr}_j \circ B\bar{x} = \bar{x}$ for any $\bar{x} \in \operatorname{Gr}_j(t)$. Our next goal is to extend the maps $A_t$, defined above on the spaces $\bigoplus_{i=1}^{k}\mathcal{P}_{\sigma_i}(t)$, to symplectic splittings of the filtrations $\{\Lambda^{(j)}(t)\}_{j\in\mathbb{Z}}$ in a canonical way. Denote $\operatorname{Gr}_-(t) = \bigoplus_{j\le 0}\operatorname{Gr}_j(t)$ and $\operatorname{Gr}_+(t) = \bigoplus_{j>0}\operatorname{Gr}_j(t)$. First, we can extend $A_t$ to $\operatorname{Gr}_-(t) \oplus \operatorname{Gr}_1(t)$, using recursively the following formula:

$$A_t(\delta s(t))) = \frac{d}{dt}A_t(s(t)) \quad \forall\, s(t) \in \operatorname{Gr}_-(t) \text{ such that } s(t) \text{ is a horizontal section in } \operatorname{Gr} W. \tag{2}$$

Note that from the definition of the connection on the bundle $\operatorname{Gr} W$ it follows that this extension of $A_t$ is compatible with the restrictions of the symplectic forms to $\operatorname{Gr}_-(t) \oplus \operatorname{Gr}_1(t)$ and $\Lambda^{(1)}(t)$. Further, recall that the symplectic form $\bar{\omega}$ defines the natural identification between $\operatorname{Gr}_-(t)$ and $\operatorname{Gr}_+(t)^*$: $v \in \operatorname{Gr}_-(t) \sim \bar{\omega}(\cdot, v) \in \operatorname{Gr}_+(t)^*$. If for any $t$ a symplectic splitting $A_t$ of the filtration $\{\Lambda^{(j)}(t)\}_{j\in\mathbb{Z}}$ satisfying (2) is chosen, then there exists a family of self-adjoint maps $R_t : \operatorname{Gr}_+(t) \mapsto \operatorname{Gr}_+(t)^*$ ($\sim \operatorname{Gr}_-(t)$) such that

$$\frac{d}{dt}A_t(s(t)) = A_t\big(\delta s(t) + R_t s(t)\big) \quad \forall\, s(t) \in \operatorname{Gr}_+(t) \text{ such that } s(t) \text{ is a horizontal section in } \operatorname{Gr} W. \tag{3}$$

The canonical symplectic splitting of the filtration $\{\Lambda^{(j)}(t)\}_{j\in\mathbb{Z}}$ will be chosen by imposing certain conditions on the map $R_t$ from (3). Let $\overline{\Delta}$ be the diagram obtained from $\Delta$ by the reflection w.r.t. its left edge. Denote by $l$ the left shift on the diagram $\Delta \cup \overline{\Delta}$ and by $\bar{a}$ the box of $\Delta \cup \overline{\Delta}$, obtained from the box $a$ by the reflection of $a$ w.r.t. the left edge of $\Delta$. Let $\mathcal{P}_{l^j(\sigma_i)}(t) = \delta^j\big(\mathcal{P}_{\sigma_i}(t)\big)$ for any $1 \le j \le 2p_i - 1$. Then $\operatorname{Gr}_-(t) = \bigoplus_{a\in\Delta}\mathcal{P}_a(t)$. For any box $b \in \Delta$ let $\pi_b$ be the canonical projection from $\operatorname{Gr}_-(t) = \bigoplus_{a\in\Delta}\mathcal{P}_a(t)$ to $\mathcal{P}_b(t)$. From the condition (G) it follows that for any box $a \in \Delta$ the restriction of the symplectic form $\bar{\omega}$ to $\mathcal{P}_a(t) \oplus \mathcal{P}_{\bar{a}}(t)$ is nondegenerate, hence identifies $\mathcal{P}_a(t)$ and $\mathcal{P}_{\bar{a}}(t)^*$. Given any self-adjoint map $R : \operatorname{Gr}_+(t) \mapsto \operatorname{Gr}_+(t)^*$ ($\sim \operatorname{Gr}_-(t)$) and a pair of boxes $(a,b)$ of $\Delta$ denote by $R(a,b)$ the following map from $\mathcal{P}_{\bar{a}}(t)$ to $\mathcal{P}_b(t) \sim P_{\bar{b}}(t)^*$: $R(a,b) = \pi_b \circ R\,|_{\mathcal{P}_{\bar{a}}}$. Since the map $R$ is self-adjoint, one has $R(b,a)^* = R(a,b)$ for any $(a,b) \in \Delta \times \Delta$.

Further, denote by $a_i$ the first box of the $i$th row of diagram $\Delta$ and by $r : \Delta\backslash\{\sigma_i\}_{i=1}^{d} \mapsto \Delta$ the right shift on the diagram $\Delta$. For any pair of integers $(i,j)$ such that $1 \le j < i \le k$ consider the following sequence of pairs of boxes

$$\begin{aligned} &\big(a_j, a_i\big), \big(a_j, r(a_i)\big), \big(r(a_j), r(a_i)\big), \big(r(a_j), r^2(a_i)\big), \ldots, \big(r^{p_i-1}(a_j), r^{p_i-1}(a_i)\big),\\ &\big(r^{p_i}(a_j), r^{p_i-1}(a_i)\big), \ldots, \big(r^{p_j-1}(a_j), r^{p_i-1}(a_i)\big). \end{aligned} \tag{4}$$

**Definition** 1 *A self-adjoint map $R : \operatorname{Gr}_+(t) \mapsto \operatorname{Gr}_+(t)^*(\sim \operatorname{Gr}_-(t))$ is called normal, if the following two conditions hold:*

*(i) Among all linear maps $R(a,b)$, where the box $b$ is not higher than the box $a$ in the diagram $\Delta$, the only possible nonzero maps are the following: the maps $R(a,a)$ for all $a \in \Delta$, the maps $R\big(a, r(a)\big)$, $R\big(r(a), a\big)$ for all $a \notin \{\sigma_i\}_{i=1}^{k}$, and the maps, corresponding to the pairs, which appear in the sequence* (4)*, starting with the $(p_j - p_i)$th pair in the sequence* (4)*, for any $1 \le j < i \le k$;*

*(ii) For any $a \notin \{\sigma_i\}_{i=1}^{k}$ the map $\delta \circ R(a, r(a)) : \mathcal{P}_{\bar{a}}(t) \mapsto \mathcal{P}_a(t) \sim (\mathcal{P}_{\bar{a}}(t)^*)$ is anti-self-adjoint.*

A pair of boxes $(a,b)$ of $\Delta$ is called *essential* if the maps $R(a,b)$ corresponding to a normal map $R : \operatorname{Gr}_+(t) \mapsto \operatorname{Gr}_+(t)^*$ are not necessarily equal to zero. A family of symplectic splittings $A_t : \operatorname{Gr} W(t) \mapsto W$ of the filtrations $\{\Lambda^{(j)}(t)\}_{j\in\mathbb{Z}}$ is said to be *normal*, if it satisfies relation (2) and (3), where all maps $R_t$ are normal.

**Theorem 1** *For any curve $\Lambda : [0,T] \mapsto L(W)$ having the Young diagram $D$ and satisfying condition (G) there exists a unique normal family of symplectic splittings $A_t : \operatorname{Gr} W(t) \mapsto W$ of the filtrations $\{\Lambda^{(j)}(t)\}_{j\in\mathbb{Z}}$.*

Now we give several consequences of Theorem 1. In what follows $A_t$ denotes the normal family of symplectic splittings of the filtrations $\{\Lambda^{(j)}(t)\}_{j\in\mathbb{Z}}$. The map $R_t : \operatorname{Gr}_+(t) \mapsto \operatorname{Gr}_+(t)^*$ corresponding to the normal symplectic splitting $A_t$ is called *the curvature map associated with the curve $\Lambda$ at the point $t$* and the corresponding map $R_t(a,b)$ is called the *$(a,b)$-curvature maps of the curve $\Lambda$ at $t$* for any essential pair $(a,b)$. The canonical connection of the bundle $\operatorname{Gr} W$ defines parallel transport isomorphisms between the symbols $\big(\bigoplus_{j\in\mathbb{Z}} \operatorname{Gr}_j(t), \delta\big)$ and the symbol $\big(\bigoplus_{j\in\mathbb{Z}} \operatorname{Gr}_j(0), \delta\big)$. Using these isomorphisms, the maps $A_t$ and $R_t$ can be considered as elements of $\operatorname{Hom}(\operatorname{Gr} W(0), W)$ and $\operatorname{Hom}(\operatorname{Gr}_+(0), \operatorname{Gr}_+(0)^*)$ respectively. The group $G_{\mathrm{Aut}}$ of automorphisms of the symbol $\big(\bigoplus_{j\in\mathbb{Z}} \operatorname{Gr}_j(0), \delta\big)$ is isomorphic to $O(r_1 - r_1^-, r_1^-) \times \ldots \times O(r_k - r_k^-, r_k^-)$. As a direct consequence of Theorem 1 and the "structural equations" (2), (3), we have the following

**Theorem 2** *Given a family of normal maps $R_t : \operatorname{Gr}_+(0) \mapsto \operatorname{Gr}_+(0)^*$ there exists a curve $\Lambda : [0,T] \mapsto L(W)$ with symbol $\big(\bigoplus_{j\in\mathbb{Z}} \operatorname{Gr}_j(0), \delta\big)$ at 0 and curvature map equal to $R_t$ for any $t \in [0,T]$. The curve $\widetilde{\Lambda} : [0,T] \mapsto L(W)$ with the same symbol $\big(\bigoplus_{j\in\mathbb{Z}} \operatorname{Gr}_j(0), \delta\big)$ at 0 is symplectic equivalent to the curve $\Lambda$ iff there exists an automorphism $g \in G_{\mathrm{Aut}}$ such that the curvature map $\widetilde{R}_t$ of the curve $\widetilde{\Lambda}$ satisfies $\widetilde{R}_t = g^* \circ R_t \circ g$ for any $t \in [0,T]$.*

If $r_i = 1$ for all $1 \le i \le k$ (or, equivalently, all rows of the diagram $D$ have different length), then all $(a,b)$-curvature maps are determined by scalar functions of $t$, which will be called shortly *$(a,b)$-curvatures of* $\Lambda$. In this case the group of automorphisms of the symbol $\big(\bigoplus_{j\in\mathbb{Z}} \operatorname{Gr}_j(0), \delta\big)$ is discrete ($\sim (\mathbb{Z}_2)^k$). It implies the following

**Corollary 1** *If all rows of the diagram $D$ have different length, then given a tuple of smooth functions $\{\rho_{a,b}(t) : (a,b) \in \Delta \times \Delta, (a,b)$ is an essential pair$\}$ and a tuple of $\{r_i^-\}_{i=1}^k$ , where $r_i^- \in \{0,1\}$, there exists the unique, up to a symplectic transformation, curve $\Lambda : [0,T] \mapsto L(W)$ having the Young diagram $D$ and satisfying condition (G) such that for any $t$ and any essential pair $(a,b)$ its $(a,b)$-curvature at $t$ coincides with $\rho_{a,b}(t)$ and its $i$th negative index is equal to $r_i^-$ for any $1 \le i \le k$.*

The normal family of symplectic splittings $A_t : \operatorname{Gr} W(0) \mapsto W$ produces a principal $G_{\mathrm{Aut}}$-bundles over $[0,T]$ of symplectic (Darboux) frames in $W$ endowed with a canonical principal connection (or $G_{\mathrm{Aut}}$-bundles of moving symplectic frames). For this fix a symplectic frame in $\operatorname{Gr} W(0)$ compatible with the grading of $\operatorname{Gr} W(0)$ and take the image under $A_t$ of all frames in the orbit of the chosen frame w.r.t. the action of $G_{\mathrm{Aut}}$. Finally, let $V_a(t) = A_t\big(P_a(0)\big)$ for any $a \in \Delta$. Then we have the canonical splitting of the subspaces $\Lambda(t)$: $\Lambda(t) = \bigoplus_{a\in\Delta} V_a$. The curve $\Lambda^{\mathrm{trans}}(t) = A_t(Gr^+(0))$ is said to be the *normal complementary curve to the curve* $\Lambda(t)$. For an essential pair $(a,b) \in \Delta \times \Delta$, where $a$ belongs to $j$th column of $\Delta$, define $\mathfrak{R}_t(a,b) \in \operatorname{Hom}(V_a, V_b)$ as follows: $\mathfrak{R}_t(a,b) = A_t \circ R_t(a,b) \circ \delta^{2j-1} \circ A_t^{-1}|_{V_a}$. In the next section by $(a,b)$-curvature map we will mean $\mathfrak{R}_t(a,b)$ instead of $R_t(a,b)$.

## 3. Consequences for geometric structures

Let $\mathcal{V}$ be a geometric structure on a manifold $M$, i.e. a submanifold $\mathcal{V} \subset TM$ transversal to the fibers. Let $\mathcal{V}_q = \mathcal{V} \cap T_qM$. Fix $c$ in $\mathbb{R}$ and define the "dual" $H_q^c$ of $\mathcal{V}_q$ by $H_q^c = \{\lambda \in T_q^*M : \exists v \in \mathcal{V}_q,\ \langle \lambda, v\rangle = c,\ \langle \lambda, T_v\mathcal{V}_q\rangle = 0\}$ and let $H^c$ be the union of the $H_q^c$. There are two essentially different cases: $c = 1$ and $c = 0$. For simplicity assume that $H^c$ is a codimension 1 submanifold of $T^*M$ and transversal to the $T_q^*M$. Let $\pi : T^*M \mapsto M$ be the canonical projection, $\varsigma$ be the canonical Liouville 1-form, and $\bar{\omega} = d\varsigma$ be the standard symplectic structure on $T^*M$. The restriction $\bar{\omega}|_{H^c}$ of $\bar{\omega}$ to $H^c$ has one-dimensional kernels at any point, which are transversal to the fibers of $T^*M$. They form a line distribution in $H^c$ and define the *characteristic 1-foliation* $\mathcal{C}$ of $H^c$. The leaves of this foliation are the *characteristic curves* of $\sigma|_{H^c}$.

Suppose that $\gamma$ is a segment of a characteristic curve and $O_\gamma$ is a neighborhood of $\gamma$ in $H^c$ such that $N = O_\gamma/(\mathcal{C}|_{O_\gamma})$ is a well-defined smooth manifold. The quotient manifold $N$ has a symplectic structure $\omega$ induced by $\bar{\omega}|_{H^c}$. Let $\phi : O_\gamma \to N$ be the canonical map; then $\phi(H_q^c \cap O_\gamma)$, $q \in M$, are Lagrangian submanifolds in $N$. Set $J_\gamma(\lambda) = \phi_*(T_\lambda H^c_{\pi(\lambda)})$, for any $\lambda \in \gamma$. *The Jacobi curve of the characteristic curve* $\gamma$ is the map $\lambda \mapsto J_\gamma(\lambda)$ from $\gamma$ to $L(T_\gamma N)$. Jacobi curves are invariants of the hypersurface $H^c \subset T^*M$ and hence of the original geometric structure $\mathcal{V}$. So, any invariant of the Jacobi curves w.r.t. the natural action of $Sp(T_\gamma N)$ produces a function on $H^c$ intrinsically related to the original geometric structure $\mathcal{V}$. It turns out ([1, Introduction]) that the tangent vectors to the Jacobi curve $J_\gamma$ at a point $\Lambda$ are equivalent (under linear substitutions of variables in the correspondent quadratic forms) to the "second fundamental form" of the hypersurface $H^c_{\pi(\lambda)} \subset T^*_{\pi(\lambda)}M$ at the point $\lambda$, while the rank of the latter is not greater than $\dim \mathcal{V}_{\pi(\lambda)}$.

Assume that the germ of the curve $J_\gamma$ at $\lambda$ satisfies condition (G) and has the Young diagram $D$ with the reduced diagram $\Delta$. Such $\lambda$ is said to be *$D$-regular.* If $c \neq 0$, then the curve $\gamma$ (and therefore the Jacobi curve $J_\gamma$) is endowed with a *canonical parameterization*, up to a shift. Indeed, in this case the Liouville form does not vanish on the tangent lines to $\gamma$ and the *canonical parametrization* is defined by the rule $\varsigma(\dot\gamma) = c$. Let $J_\gamma(\lambda) = \bigoplus_{a\in\Delta} \widetilde{\mathfrak{A}}_a(\lambda)$ be the canonical splitting of the subspace $J_\gamma(\lambda)$ (w.r.t. the canonically parametrized curve $J_\gamma$). Set $\mathfrak{A}_a(\lambda) = \phi_*^{-1}\big(\widetilde{\mathfrak{A}}_a(\lambda)\big) \cap T_\lambda H^c_{\pi(\lambda)}$ Taking into account that $\phi_*$ establishes an isomorphism between $T_\lambda H^c_{\pi(\lambda)}$ and $J_\gamma(\lambda)$, we get the following *canonical splitting of the tangent space $T_\lambda\big(T^*_{\pi(\lambda)}M\big)$ to the fiber of $T^*M$ at $\lambda$*: $T_\lambda T^*_{\pi(\lambda)}M = \bigoplus_{a\in\Delta} \widetilde{\mathfrak{A}}_a(\lambda) \oplus \operatorname{span}\{\epsilon(\lambda)\}$, where $\epsilon$ is the Euler field of $T^*M$ (the infinitesimal generator of the homotheties of the fibers of $T^*M$). Besides, each subspace $\mathfrak{A}_a(\lambda)$ is endowed with the canonical Euclidean structure and the corresponding curvature maps between the subspaces of the splitting are intrinsically related to the geometric structure $\mathcal{V}$. Further, let $\mathrm{Hor}(\lambda) = (\phi_*)^{-1}\big(J_\gamma^{\mathrm{trans}}(\lambda)\big)$ , where $J_\gamma^{\mathrm{trans}}(\lambda)$ is the subspace corresponding to the normal complementary curve to the Jacobi curve $J_\gamma$ at the point $\lambda$. Then $\mathrm{Hor}(\lambda)$ is transversal to $T_\lambda\big(T^*_{\pi(\lambda)}M\big)$. So, if for some diagram $D$ the set $U$ of its $D$-regular points is open in $T^*M\backslash H^0$, then the distribution of "horizontal" subspaces $\mathrm{Hor}(\lambda)$ defines a *connection on $U \subset T^*M$, canonically associated with the geometric structure $\mathcal{V}$*. Note also that in the case of sub-Riemannian structures the set $H^1_q$ is of the form $Q = 1$, where $Q$ is a positive semidefinite quadratic form. In this case assume that $J_\gamma^{(p)}(\lambda) = T_\gamma N$ for some $\lambda$ and $p$. Then in any neighborhood $U$ of $\lambda$ in $T^*M$ there is an nonempty open set $\mathcal{O} \subset U$ of $D$-regular points for some Young diagram $D$. Moreover, for any $q \in \pi(\mathcal{O})$ the set of $D$-regular points of the fiber $T^*_qM$ is a nonempty Zariski open subset. Besides, the canonical splitting, the curvature maps, and the canonical connection above depend rationally on points of the fiber $T^*_qM$. Finally note that in the case of a Riemannian metric the canonical connection above coincides with the Levi-Civita connection, the reduced Young diagram of Jacobi curves consists of only one box, and the Riemannian curvature tensor can be recovered from the corresponding curvature map.

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## References

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