On feedback classification of control-affine systems with one and two-dimensional inputs

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Abstract

The paper is devoted to the local classification of generic control-affine systems on an n-dimensional manifold with scalar input for any $n \geq 4$ or with two inputs for $n = 4$ and $n = 5$, up to state-feedback transformations, preserving the affine structure (in C^∞ category for $n = 4$ and C^ω category for $n \geq 5$). First using the Poincare series of moduli numbers we introduce the intrinsic numbers of functional moduli of each prescribed number of variables on which a classification problem depends. In order to classify affine systems with scalar input we associate with such a system the canonical frame by normalizing some structural functions in a commutative relation of the vector fields, which define our control system. Then, using this canonical frame, we introduce the canonical coordinates and find a complete system of state-feedback invariants of the system. It also gives automatically the micro-local (i.e. local in state-input space) classification of the generic non-affine n-dimensional control system with scalar input for $n \geq 3$ (in C^∞ category for $n = 3$ and in C^ω category for $n \geq 4$). Further we show how the problem of feedback-equivalence of affine systems with two-dimensional input in state space of dimensions 4 and 5 can be reduced to the same problem for affine systems with scalar input. In order to make this reduction we distinguish the subsystem of our control system, consisting of the directions of all extremals in dimension 4 and all abnormal extremals in dimension 5 of the time optimal problem, defined by the original control system. In each classification problem under consideration we find the intrinsic numbers of functional moduli of each prescribed number of variables according to its Poincare series.

Key words: State-feedback equivalence, control-affine systems, Poincare series, extremals.

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1 Introduction

In the paper for the convenience of the presentation all objects are C^∞ without special mentioning, although all constructions and some statements remain valid in an obvious way also in C^k category for an appropriate finite k. On the other hand, some of the statements are known to be true only in the real analytic category, which will be indicated explicitly.

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Let M be an n-dimensional manifold, $f_0, f_1, \dots, f_r$ be vector fields on M, $r < n$. Consider the following control-affine system with r inputs on M

$$\dot q = f_0(q) + \sum_{k=1}^{r} u_r\, f_r(q), \quad q \in M, \quad u_1 \dots u_r \in \mathbb{R}. \tag{1.1}$$

We say that a system of the type (1.1) is an (r, n) control-affine system. We also assume that at given point q_0

$$\dim \operatorname{span}\big(f_0(q_0), f_1(q_0), \dots, f_r(q_0)\big) = r + 1. \tag{1.2}$$

Consider the group FB_{q_0} of state-feedback transformations, preserving an affine structure and the point q_0, i.e. transformations of the type

$$\left\{ \begin{array}{l} q = \Phi(\tilde q) \\ u = \mathcal{B}(\tilde q)\tilde u + \mathcal{A}(\tilde q) \\ q_0 = \Phi(q_0), \end{array} \right. , \quad u = \begin{pmatrix} u_1 \\ \vdots \\ u_r \end{pmatrix}, \tag{1.3}$$

where Φ is a diffeomorphism in a neighborhood of q_0, $\mathcal{A}(q) \in \mathbb{R}^r$, $\mathcal{B}(q)$ is a $r \times r$-matrix, $\det \mathcal{B}(q_0) \neq 0$. This group of transformations acts naturally on the set of germs of systems of the type (1.1) and defines the equivalence relation, called *state-feedback equivalence.* The natural question is when two germs of the systems of the type (1.1) are state-feedback equivalent.

Let us roughly estimate the "number of parameters" in the considered equivalence problem. The set of r-dimensional affine subspaces in $\mathbb{R}^n$ forms $(r+1)(n-r)$-dimensional manifold. Therefore, if the coordinates on M are fixed, then the control system of the type (1.1) can be defined by $(r+1)(n-r)$ functions of n variables. The group of the coordinate changes on M is parameterized by n functions of n variables. So, by a coordinate change one can "normalize", in general, only n functions among those $(r+1)(n-r)$ functions, defining our control system. Thus we may expect that the set of orbits of generic germs of systems (1.1) at $q_0 \in M$ w.r.t. the action of the group of transformations of the type (1.3) can be parameterized by $(r+1)(n-r) - n = r(n-r-1)$ arbitrary germs of functions of n variables and a number of germs of functions, depending on less than n variables (see the next section for the discussion about the number of these additional functional invariants).

According to the last estimate the only case when the functional parameters (also called functional moduli) are not expected in the parameterization of the set of orbits of generic germs of systems (1.1) is the case $r = n-1$ (i.e., corank 1 control-affine systems). In this case under assumption (1.2) there is the natural one-to-one correspondence between the set of control-affine systems, up to feedback transformations, and the set of differential 1-forms on the ambient manifold: to any affine system (1.1) one can assign a unique differential 1-form ω such that $\omega(f_i) = 0$ for $i = 1, \dots n-1$ and $\omega(f_0) = 1$. So, the state-feedback classification of corank 1 control-affine systems satisfying (1.2) is equivalent to the well-known classification of differential 1-forms w.r.t. the action of the group of diffeomorphisms (see, for example, [8], section 3 and Appendix C there). In particular, all germs of (1.1) such that the underlying vector distribution $\operatorname{span}(f_1, \dots, f_{n-1})$ is contact for odd n or quasi-contact for even n (which is generic assumption) are state-feedback equivalent to the control-affine system, corresponding to the classical Darboux model (note also that in [8] normal forms for codimension 1 singularities are given too). In the context of corank 1 control-affine systems it is not worse also to mention the works [3], [6], [9], where the case $f(q_0) \in \operatorname{span}\big(f_1(q_0), \dots, f_{n-1}(q_0)\big)$ was treated.

The case $r = 1$, $n = 3$ (the smallest dimensions, when the functional parameters appear) was treated in [1] (section 3, Proposition 3.2 there). In particular, it was shown that the set of orbits of generic germs of the systems w.r.t. the action of the group of transformations of the type (1.3) can be parametrized by one arbitrary function of 3 variables, 2 arbitrary functions of two variable , and the discrete invariant from the set $\{-1, 1\}$.

Remark 1 Actually, in Proposition 3.2 of [1] the two functions of two variables satisfy certain conditions on coordinate subspaces of some special coordinates, which are canonical up to some reflections, but instead of these functions one can take their appropriate partial derivatives, which are already arbitrary. The functions of the parameterization are state-feedback invariants up to some reflections in the coordinates.$\Box$

In the present paper we make a classification of generic germs of systems of the type (1.1), up to state-feedback equivalence, in the following cases

1. $r = 1$, $n = 4$;
2. $r = 1$, $n \geq 5$ in the real analytic category;
3. $r = 2$, $n = 4$;
4. $r = 2$, $n = 5$ in the real analytic category.

In general, statements of the kind "the classification problem depends on the tuple of functional invariants, consisting of certain number of functions of each number of variables" need to be clarified: these numbers could be changed rather arbitrary by mixing, combining or separating the formal Taylor series of this functional invariants without losing any information (at least if we work in the category of formal Taylor series or in the real analytic category). In [2] (section 1 there) it was proposed to use the so-called *Poincare series of the moduli numbers of the classification problem* in order to determine intrinsically the number of functions of each prescribed number of variables, on which some classification problem depends. In section 2 below, using the Poincare series, we give a canonical selection of these numbers. Throughout this section we demonstrate all our notions and constructions on the problem of classification of germs of Riemannian metrics on a two-dimensional manifold. The way the canonical parameterization is obtained indicates the presence of an interesting algebraic structure on the set of all tuples of fundamental invariants parameterizing given classification problem. For the moment, this algebraic structure remains hidden and needs further research.

In the case of scalar input our method of the classification is similar to the procedure, used in [1] for the case $n = 3$ and it is described in section 3. It consists basically of the following two steps: first for any control system, satisfying some genericity assumptions, we construct the canonical frame by normalizing some structural functions in a commutative relation of the vector fields, which define our control-affine systems; then, using this canonical frame, we introduce the canonical coordinates and find the complete system of state-feedback invariants of the system.

Besides, to any control system

$$\dot{y} = \mathcal{F}(y, v), \quad y \in S, v \in V \tag{1.4}$$

on an m-dimensional manifold S (the state space) with one-dimensional control space V one can assign the following control-affine system on the $(m+1)$-dimensional state-space $S \times V$.

$$\begin{cases} \dot{y} = \mathcal{F}(y, v) \\ \dot{v} = u \end{cases}, \quad v \in \mathbb{R} \tag{1.5}$$

(here we look on u as on a new state variable, v is the new control, $f_0 = (\mathcal{F}(y,v),0)^T$ and $f_1 = (0,1)^T$ in the notations of (1.1)). It turns out that having the local classification of generic $m+1$-dimensional control-affine systems with scalar input, one gets also the micro-local (i.e. local in state-input space) classification of the generic m-dimensional control system (see, Remark 6 at the end of section 3).

Further, in section 4, we show that the problem of state-feedback classification of the control-affine systems with two-dimensional input in dimensions 4 and 5 can be reduced to the previous problem for the control-affine systems with scalar input in the same dimensions. In order to make this reduction we distinguish the subsystem, consisting of the directions of all extremals in dimension 4 and all abnormal extremals in dimension 5 of the time optimal problem, defined by the original control system.

In each classification problem under consideration we find the intrinsic numbers of functional moduli of each prescribed number of variables according to its Poincare series.

Finally note that the problem, considered here, is different from one, considered in the paper [7], which has the similar title. In the mentioned paper the authors study germs of n-dimensional control-affine system with scalar input at an equilibrium point q_0, i.e. when $f_0(q_0) \in \{\mathbb{R}f_1(q_0)\}$. Their method is a generalization of technique, developed in [5] and [4], which is similar to classical Poincare-Dulac procedure for normalization of vector fields near a stationary point. Therefore in the method of [7] it is crucial that q_0 is an equilibrium point. Here we classify the control-affine systems with scalar control near non-equilibrium point, which seems very natural in view of the fact that this classification, except the case $n=2$, a priori contains functional moduli. Besides, the feedback invariants, constructed here for generic germs could be used also for the problem with equilibrium points by passing to the limit.

2 Poincare series and the intrinsic number of functional invariants.

We start with some terminology. Let M be a smooth manifold. Fix a point $q_0 \in M$. Consider a set $\mathcal{O}$ of germs at q_0 of smooth objects on M (for example, Riemannian metrics, vector distributions, control-affine systems) such that the group of local diffeomorphisms Diff_{q_0}, preserving the point q_0, acts naturally on it. This action defines the equivalence relation on $\mathcal{O}$.

Denote by $J^k(\mathcal{O})$ the space of all k-jets at q_0 of objects from the set $\mathcal{O}$. We say that the set $\widetilde{\mathcal{O}} \subset \mathcal{O}$ is *a generic subset* of $\mathcal{O}$ if there exists an integer $k \geq 0$ and a Zariski open set U in $J^k(\mathcal{O})$ such that

$$\widetilde{\mathcal{O}} = \{\mathfrak{b} \in \mathcal{O} : k\text{-jet of } \mathfrak{b} \text{ belongs to } U\}.$$

By classification problem on $\mathcal{O}$ we mean the problem to find a system of fundamental invariants for objects from some generic subset of $\mathcal{O}$ such that two generic objects are equivalent if and only if they have the same systems of fundamental invariants.

Let $\widetilde{\mathcal{O}}$ be a generic subset of $\mathcal{O}$, which is invariant w.r.t. the action of the group Diff_{q_0}.

Definition 1 *A mapping F from the set $\widetilde{\mathcal{O}}$ to the set $C_0^\infty(\mathbb{R}^l,\mathbb{R})$ of germs at 0 of smooth functions in $\mathbb{R}^l$, which is invariant w.r.t. the action of the group Diff_{q_0} on $\widetilde{\mathcal{O}}$, is called a functional invariant of l variables of a generic subset of objects from $\mathcal{O}$ (or shortly the functional invariant of $\mathcal{O}$).*

When the object $\mathfrak{b} \in \widetilde{\mathcal{O}}$ is fixed, we will mean by the functional invariant also the value of the mapping F at $\mathfrak{b}$. We will denote this germ of function by the same letter F without special mentioning.

Let $\mathrm{Orb}(\widetilde{\mathcal{O}})$ be the set of orbits of $\widetilde{\mathcal{O}}$ w.r.t. the action of Diff_{q_0}. Then any functional invariants $F : \widetilde{\mathcal{O}} \mapsto C_0^\infty(\mathbb{R}^l, \mathbb{R})$ induces the mapping $\widehat{F} : \mathrm{Orb}(\widetilde{\mathcal{O}}) \mapsto C_0^\infty(\mathbb{R}^l, \mathbb{R})$ in the obvious way.

Definition 2 *Let $\{F_i\}_{i=1}^s$ be the tuple of functional invariants of $\mathcal{O}$ defined on $\widetilde{\mathcal{O}}$, where each F_i is a functional invariant of l_i variables. We say that the tuple $\{F_i\}_{i=1}^s$ defines a parameterization of the classification problem on $\mathcal{O}$ if the mapping*

$$(\widehat{F}_1, \dots, \widehat{F}_s) : \mathrm{Orb}(\widetilde{\mathcal{O}}) \mapsto C_0^\infty(\mathbb{R}^{l_1}, \mathbb{R}) \times \dots \times C_0^\infty(\mathbb{R}^{l_s}, \mathbb{R})$$

is injective and has open image in $C_0^\infty(\mathbb{R}^{l_1}, \mathbb{R}) \times \dots \times C_0^\infty(\mathbb{R}^{l_s}, \mathbb{R})$.

Example 1 Let $\mathcal{O}_1$ be a set of germs of Riemannian metrics at a point q_0 on an oriented two-dimensional manifold M. Given a germ of metric G let K_G be its Gaussian curvature. Let $\widetilde{\mathcal{O}}_1$ be the set of all germs G of Riemannian metrics at q_0 such that $dK_G(q_0) \neq 0$. Obviously, $\widetilde{\mathcal{O}}_1$ is generic subset of $\mathcal{O}_1$. For any $G \in \mathcal{O}_1$ consider the geodesic polar coordinates (r, φ) centered at q_0 and agreed with the orientation such that the vector $\mathrm{grad}\, K_G(q_0)$ is in direction of the ray $\{\varphi = 0\}$. We will call these coordinates *the canonical polar coordinates of* G. The corresponding Cartesian coordinates (x_1, x_2) will be called *the canonical coordinates of the metric G at q_0*. Then the mapping $\mathfrak{K} : \widetilde{\mathcal{O}}_1 \mapsto C_0^\infty(\mathbb{R}^2, \mathbb{R})$ such that

$$\mathfrak{K}(G) = K_G(x_1, x_2) \tag{2.1}$$

is a functional invariant of two variables of $\mathcal{O}_1$.

Now let us construct the parametrization of the classification problem on $\mathcal{O}_1$. By construction the image of $\mathfrak{K}$ lies in the set

$$\mathcal{N} = \{f \in C_0^\infty(\mathbb{R}^2, \mathbb{R}) : f_{x_2}(0,0) = 0, f_{x_1}(0,0) > 0\}.$$

Actually, $\mathfrak{K}$ defines the one-to-one correspondence between the set $\mathrm{Orb}\big(\widetilde{\mathcal{O}}_1\big)$ of orbits of $\widetilde{\mathcal{O}}_1$ w.r.t. the action of Diff_{q_0} and the set $\mathcal{N}$. Indeed, for any $f \in \mathcal{N}$ take the metric G, which can be written in some polar coordinates centered at q_0 in the following way:

$$G = dr^2 + \big(B(r, \varphi)\big)^2 d\varphi^2, \tag{2.2}$$

where

$$\begin{cases} \dfrac{\partial^2}{\partial r^2} B + f(r\cos\varphi, r\sin\varphi) B = 0 \\ B(0, \phi) = 0, \quad \dfrac{\partial}{\partial r} B(0, \phi) = 1 \end{cases} \tag{2.3}$$

Then $\mathfrak{K}(G) = f$ and any germ of metric $\overline{G}$ at q_0 such that $\mathfrak{K}(\overline{G}) = f$ is isometric to G.

However $\mathfrak{K}$ does not define the parametrization of the considered classification problem in the sense of Definition 2, because the set $\mathcal{N}$ is not open subset of $C_0^\infty(\mathbb{R}^2, \mathbb{R})$. But instead of $\mathfrak{K}$ we can consider the following two functional invariants of one variables and one functional invariant of two variables:

$$\mathfrak{K}_1(G) = K_G(x_1, 0), \quad \mathfrak{K}_2(G) = \frac{\partial^2}{\partial x_2^2} K_G(0, x_2), \quad \mathfrak{K}_3(G) = \frac{\partial^2}{\partial x_1 \partial x_2} K_G(x_1, x_2), \tag{2.4}$$

where (x_1, x_2) are the canonical coordinates of the metric G at q_0. The image of the mapping $(\mathfrak{K}_1, \mathfrak{K}_2, \mathfrak{K}_3)$ is open in $C_0^\infty(\mathbb{R}, \mathbb{R}) \times C_0^\infty(\mathbb{R}, \mathbb{R}) \times C_0^\infty(\mathbb{R}^2, \mathbb{R})$, namely it is equal to

$$\mathcal{N}_1 = \{f_1(x_1), f_2(x_2), f_3(x_1, x_2) \in C_0^\infty(\mathbb{R}, \mathbb{R}) \times C_0^\infty(\mathbb{R}, \mathbb{R}) \times C_0^\infty(\mathbb{R}^2, \mathbb{R}) : f_1'(0) > 0\}$$

Besides, the original functional invariant $\mathfrak{K}$ can be uniquely recovered from the tuple $(\mathfrak{K}_1, \mathfrak{K}_2, \mathfrak{K}_3)$. So, the tuple $(\mathfrak{K}_1, \mathfrak{K}_2, \mathfrak{K}_3)$ defines the one-to-one correspondence between the set $\mathrm{Orb}(\tilde{\mathcal{O}}_1)$ and the set $\mathcal{N}_1$. Hence this tuple defines the parametrization of the considered classification problem in the sense of Definition 2. □

Now let us describe the Poincare series of the moduli numbers of the classification problem. The action of the group Diff_{q_0} induces the action $\mathcal{A}_k$ of some finite dimensional Lie group G_k on the space $J^k(\mathcal{O})$ for any integer $k \geq 0$. So , $\mathcal{A}_k$ is a mapping from $G_k \times J^k(\mathcal{O})$ to $J^k(\mathcal{O})$. Given any $\mathfrak{b} \in J^k(\mathcal{O})$ let $\mathcal{A}_k^{\mathfrak{b}}$ be the mapping from G_k to $J^k(\mathcal{O})$ such that $\mathcal{A}_k^{\mathfrak{b}}(\cdot) = \mathcal{A}_k(\cdot, \mathfrak{b})$. Let e_k be the identity of the group G_k. Set

$$m(k) = \dim J^k(\mathcal{O}) - \max_{b \in J^k(\mathcal{O})} \operatorname{rank} d\mathcal{A}_k^{\mathfrak{b}}(e_k) = \min_{b \in J^k(\mathcal{O})} \operatorname{corank} d\mathcal{A}_k^{\mathfrak{b}}(e_k) \tag{2.5}$$

(here $\operatorname{rank} d\mathcal{A}_k^{\mathfrak{b}}(e_k)$ and $\operatorname{corank} d\mathcal{A}_k^{\mathfrak{b}}(e_k)$ are the rank and the corank of the differential of $\mathcal{A}_k^{\mathfrak{b}}$ at e_k respectively). Roughly speaking $m(k)$ is the dimension of the space of orbits w.r.t. the last action $\mathcal{A}_k$. The number $m(k)$ is called the *moduli number of the k-jets. The Poincare series of the moduli numbers of the classification problem* (or shortly *the Poincare series of the classification problem*) is by definition the following function:

$$M(t) = \sum_{k=0}^{\infty} m(k) t^k \tag{2.6}$$

Remark 2 Since the integer-valued function $\mathfrak{b} \mapsto \operatorname{rank} d\mathcal{A}_k^{\mathfrak{b}}(e_k)$ takes its maximal value at a Zarisky open set, in (2.5) we can replace $\mathcal{O}$ by any its generic subset $\widetilde{\mathcal{O}}$. □

The Poincare series could be useful in evaluating of the number of the functional invariants of the given number of variables, on which the given classification problem depends, because of the following well-known fact: if one denotes by $j_l(k)$ the dimension of the space $J^k(\mathbb{R}^l, \mathbb{R})$ of k-jets of functions of l-variables, then the corresponding Poincare series of numbers $j_l(k)$ satisfies

$$\sum_{k=0}^{\infty} j_l(k) t^k = \frac{1}{(1-t)^{l+1}} \tag{2.7}$$

(here one uses that $j_l(k) = \frac{(l+k)!}{l!k!}$). So, if , for example, the Poincare series of some classification problem is equal to

$$M(t) = t^w \sum_{i=1}^{n} \frac{p_i}{(1-t)^{i+1}}, \tag{2.8}$$

where all p_i are nonnegative integers, then it is natural to conclude that this problem depends on the tuple, consisting of p_i functional invariants of i variables for each $1 \leq i \leq n$, while the parameter w (i.e. the order of zero of the Poincare series $M(t)$ at $t = 0$) is equal to the minimal $k \geq 0$ such that the action of the group G_k on the space of $J^k(\mathcal{O})$ has non-discrete set of orbits.

Till now nothing is known about the form of the functions $M(t)$ for general classification problems. For example, the following open question is stated in [2]: Is it true that the Poincare series of moduli numbers are rational functions in the most classification problems? In the next sections we will show by direct computations that in all classification problems 1-4 listed in Introduction it is true and moreover the function $M(t)$ has a unique pole at $t = 1$. On the other hand, in all considered cases (except the cases $r = 1$, $n = 3$ or 5) the Poincare series has no a representation of the type (2.8) with nonnegative p_i. Below we give an algorithm to extract the number of functional invariants from Poincare series also in these cases.

From now on we will suppose that the Poincare series $M(t)$ of the classification problem is a rational function with a unique pole at $t = 1$. Let w_0 be the order of zero of the function $M(t)$ at $t = 0$.

Lemma 1 *For any integers $w \geq w_0$ and l there exist a unique polynomial $R(t)$ with*

$$\deg R(t) < w - w_0 \tag{2.9}$$

and a unique rational function $Q(t)$ with the unique pole at $t = 1$ such that

$$M(t) = \frac{t^{w_0} R(t)}{(1-t)^{l+1}} + t^w Q(t). \tag{2.10}$$

Proof. Let us fix $l \in \mathbb{Z}$ and prove the existence of a representation of the type (2.10) for any $w \geq w_0$ by induction in w.

If $w = w_0$, then from the condition (2.9) it follows that $R(t) \equiv 0$. Then by definition of order of zero the function $Q(t) = \frac{1}{t^{w_0}} M(t)$ is rational with the unique pole at $t = 1$, which implies (2.10).

Now suppose that a representation of the type (2.10) exists for some $w = \bar{w}$, $\bar{w} \geq w_0$, and prove its existence for $w = \bar{w} + 1$. For this let $Q(t)$ and $R(t)$ be as in the representation (2.10) for $w = \bar{w}$. Denote by

$$Q_1(t) = \frac{1}{t}\left(Q(t) - \frac{Q(0)}{(1-t)^{l+1}}\right). \tag{2.11}$$

Then by construction Q_1 is also the rational function with the unique pole at $t = 1$. Expressing $Q(t)$ from (2.11) and substituting it into (2.10), one has

$$M(t) = \frac{t^{w_0}\big(R(t) + Q(0)t^{\bar{w}-w_0}\big)}{(1-t)^{l+1}} + t^{\bar{w}+1} Q_1(t). \tag{2.12}$$

Since $\deg\big(R(t) + Q(0)t^{\bar{w}-w_0}\big) < \bar{w} - w_0 + 1$ it implies the existence of a representation (2.10) also for $w = \bar{w} + 1$. This completes the proof by induction of the existence part of the lemma.

Now let us prove the uniqueness part. If there exists another representation of $M(t)$ of the type (2.10) with a polynomial $\bar{R}(t)$, $\deg \bar{R}(t) < w - w_0$, and a rational function $\bar{Q}(t)$ instead of $R(t)$ and $Q(t)$, then we have the following identity

$$\bar{R}(t) - R(t) = t^{w-w_0}(1-t)^{l+1}\Big(Q(t) - \bar{Q}(t)\Big).$$

It implies that the polynomial $\bar{R}(t) - R(t)$ has zero of order not less than $w - w_0$. On the other hand, by assumptions $\deg\big(R(t) - \bar{R}(t)\big) < w - w_0$, which implies that $R(t) \equiv \bar{R}(t)$ and then also $Q(t) \equiv \bar{Q}(t)$. □

We will call the representation (2.10) (with $R(t)$ satisfying (2.9)) the $(w, l)-$ *representation of the function* $M(t)$. Let N be the order of pole of $(1-t)M(t)$ at $t = 1$.

Definition 3 *The (w,l)-representation (2.10) of $M(t)$ with $R(t)$ and $Q(t)$ satisfying*

$$R(t)=\sum_{i=0}^{w-w_0-1} r_i t^i, \quad Q(t)=\sum_{j=l_1}^{N}\frac{q_j}{(1-t)^{j+1}}, \quad q_{l_1}\neq 0 \tag{2.13}$$

is called nice, if $1\le l\le N$, $l_1\ge l$, and all coefficients r_i, q_j in (2.13) are nonnegative integers.

Of course, in general a rational function $M(t)=\frac{t^{w_0}Z(t)}{(1-t)^{N+1}}$, where $Z(t)$ is a polynomial (even with integer coefficients), may not have any nice $(w,l)-$representation. But if the function $M(t)$ is the Poincare series of a classification problem, which can be parameterized by functional invariants in some reasonable way, then $M(t)$ has at least one nice representation.

To be more precise and to explain why the nice representation of the Poincare series are interesting let us introduce an additional terminology. Let F be a functional invariant of l variables of a generic subset $\widetilde{\mathcal{O}}$ of objects from $\mathcal{O}$. Denote by Π_k the natural projection from the set $\widetilde{\mathcal{O}}$ to the space $J^k(\widetilde{\mathcal{O}})$. Let $(x_1,\dots x_l)$ be the standard coordinates in $\mathbb{R}^l$. Let as before $C_0^\infty(\mathbb{R}^l,\mathbb{R})$ be the set of germs at 0 of smooth functions in $\mathbb{R}^l$ and $J^k(\mathbb{R}^l,\mathbb{R})$ be the space of k-jets of germs of these functions at 0. Denote also by π_k^l the natural projection from $C_0^\infty(\mathbb{R}^l,\mathbb{R})$ to $J^k(\mathbb{R}^l,\mathbb{R})$.

Definition 4 *The weight of the functional invariant F of l variables, defined on a generic subset $\widetilde{\mathcal{O}}\subset\mathcal{O}$, is the minimal nonnegative integer w with the following property: for any integer $k\ge w$ there exists a mapping $\mathfrak{F}_k: J^k(\widetilde{\mathcal{O}})\mapsto J^{k-w}(\mathbb{R}^l,\mathbb{R})$ such that the following diagram*

$$\begin{array}{ccc} \widetilde{\mathcal{O}} & \xrightarrow{F} & C_0^\infty(\mathbb{R}^l,\mathbb{R}) \\ {\scriptstyle \Pi_k}\downarrow & & \downarrow{\scriptstyle \pi^l_{k-w}} \\ J^k(\widetilde{\mathcal{O}}) & \xrightarrow[\mathfrak{F}_k]{} & J^{k-w}(\mathbb{R}^l,\mathbb{R}) \end{array} \tag{2.14}$$

is commutative. If such w does not exist, we will say that F has an infinite weight.

Suppose that the tuple $\{F_i\}_{i=1}^s$ defines a parameterization of the classification problem on $\mathcal{O}$ such that each F_i is a functional invariant of $l_i\ge 0$ variables and the finite weight ν_i, $\nu_i\le\nu_{i+1}$. Let $\widetilde{\mathcal{O}}$ be the common domain of definition of all invariants F_i, $1\le i\le s$. For any $k\ge 0$ set $\mu_k=\max\{i\in\{1,\dots,s\}:\nu_i\le k\}$. For given functional invariant F_i and any $k\ge\nu_i$ denote by $\mathfrak{F}_{i,k}: J^k(\widetilde{\mathcal{O}})\mapsto J^{k-\nu_i}(\mathbb{R}^{l_i},\mathbb{R})$ the corresponding mapping in the commutative diagram (2.14) for F_i. Let $\mathrm{Orb}\big(J^k(\widetilde{\mathcal{O}})\big)$ be the set of orbits of $J^k(\widetilde{\mathcal{O}})$ w.r.t. the action $\mathcal{A}_k$ of the group G_k (recall that $\mathcal{A}_k$ is the action on $J^k(\mathcal{O})$ induced by the action of the group Diff_{q_0} on $\mathcal{O}$). Then the mapping $\mathfrak{F}_{i,k}: J^k(\widetilde{\mathcal{O}})\mapsto J^{k-\nu_i}(\mathbb{R}^{l_i},\mathbb{R})$ induces the mapping $\widehat{\mathfrak{F}}_{i,k}:\mathrm{Orb}\big(J^k(\widetilde{\mathcal{O}})\big)\mapsto J^{k-\nu_i}(\mathbb{R}^{l_i},\mathbb{R})$ in the obvious way. Also let $\mathcal{U}_i$ be the image of $\widetilde{\mathcal{O}}$ under F_i. From Definition 2 it follows that the set $\mathcal{U}_i$ is open subset of $C_0^\infty(\mathbb{R}^{l_i},\mathbb{R})$. Then, according to the definition of the weight, for any $k\ge 0$ the mapping

$$(\widehat{\mathfrak{F}}_{1,k},\dots,\widehat{\mathfrak{F}}_{\mu_k,k}):\mathrm{Orb}\big(J^k(\widetilde{\mathcal{O}})\big)\mapsto J^{k-\nu_1}(\mathbb{R}^{l_1},\mathbb{R})\times\ldots\times J^{k-\nu_{\mu_k}}(\mathbb{R}^{l_{\mu_k}},\mathbb{R}) \tag{2.15}$$

is well defined and has open image equal to $\pi^{l_1}_{k-\nu_1}(\mathcal{U}_1)\times\ldots\times\pi^{l_{m_k}}_{k-\nu_{\mu_k}}(\mathcal{U}_{\mu_k})$.

Definition 5 *The parametrization, defined by the tuple $\{F_i\}_{i=1}^s$ as above, is called regular if for any $k\ge 0$ the mappings (2.15) are injective. We also say that a classification problem is regular, if it can be parameterized by a regular parameterization.*

Suppose that w_1 and w_2 are respectively the minimal and the maximal weights of functional invariants, defining some regular reparameterization, while n_1 and n_2 are respectively the minimal and maximal number of variables of these invariants. Then to such regular parameterization one can assign a $(n_2 - n_1 + 1) \times (w_2 - w_1 + 1)$ matrix P such that its (i, j)entry p_{ij} is equal to the number of the functional invariants of $i + n_1 - 1$ variables and weight $j + w_1 - 1$ in this parameterization. Directly from the definitions of regular parametrization, formula (2.7), and Remark 2 it follows that the Poincare series of the classification problem, admitting such regular parameterization, satisfies

$$M(t) = \frac{t^{w_1-1}}{(1-t)^{n_1}} \sum_{j=1}^{w_2-w_1+1} t^j \left(\sum_{i=1}^{n_2-n_1+1} \frac{p_{ij}}{(1-t)^{i+1}} \right). \tag{2.16}$$

Note that $w_1 = w_0$ and $n_2 = N$, where as before w_0 is the order of zero of $M(t)$ at $t = 0$ and N is the order of pole of $(1-t)M(t)$ at $t = 1$, but all other parameters, appearing in (2.16) could not be uniquely recovered from the Poincare series $M(t)$. In the considered situation we will say that the classification problem admits *a regular* (w_2, n_1)*-parameterization with parameterization matrix* P.

The continuation of Example 1 Consider again the situation, described in the Example 1. From the normal form (2.2)-(2.3) it follows easily that the functional invariant $\mathfrak{K}$, defined by (2.1), has the weight 2, while invariants $\mathfrak{K}_1$, $\mathfrak{K}_2$, and $\mathfrak{K}_3$, defined by (2.4) have weights 2, 4, and 4 respectively. Moreover, the tuple of invariants $(\mathfrak{K}_1, \mathfrak{K}_2, \mathfrak{K}_3)$ defines the regular $(4, 1)$-parametrization with parametrization matrix

$$P = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}. \tag{2.17}$$

The Poincare series of the considered classification problem satisfies

$$M(t) = \frac{t^2}{(1-t)^2} + t^4 \left(\frac{1}{(1-t)^3} + \frac{1}{(1-t)^2} \right) \tag{2.18}$$

□

Given some regular (w_2, n_1)-parameterization of the classification problem one can easily build a new regular (w_2, n_1)-parameterization with another parameterization matrices. Indeed, take some functional invariant F of the weight j_0, depending on i_0 variables, say $x_1, x_2, \ldots, x_{i_0}$, where $2 \le i_0 \le n_2 - n_1 + 1$ and $1 \le j_0 \le w_2 - w_1$. Let $G(x_1, \ldots, x_{i_0})$ be the function such that

$$F(x_1, \ldots, x_{i_0}) = F(x_1, \ldots, x_{i_0-1}, 0) + x_{i_0} G(x_1, \ldots, x_{i_0}). \tag{2.19}$$

Then we can obtain the new parameterization of the classification problem by replacing the functional invariant $F(x_1, \ldots, x_{i_0})$ by two functional invariants $F(x_1, \ldots, x_{i_0-1}, 0)$ and $G(x_1, \ldots, x_{i_0})$. Obviously, the first invariant has weight j_0 and depends on $i_0 - 1$ variables, while the second one has weight $j_0 + 1$ and depends on i_0 invariants. The matrix of the new parameterization is obtained from the original one by decreasing the (i_0, j_0)-entry by 1 and increasing both (i_0-1, j_0)-entry and (i_0, j_0+1) entry by 1. Such transformation on the set of $(N - n_1 + 1) \times (w_2 - w_0 + 1)$ matrices will be called *an elementary transformation*. Conversely, given two functional invariants G_1 and G_2 such that G_1 depends on $i_0 - 1$ variables say $x_1, \ldots, x_{i_0-1}$ and has the weight j_0, while G_2 depends on i_0 variables say $x_1, \ldots, x_{i_0}$ and has the weight $j_0 + 1$ (here again

$2 \le i_0 \le n_2 - n_1 + 1$ and $1 \le j_0 \le w_2 - w_1$) one can build the new parameterization by replacing the invariants G_1 and G_2 by one invariant $G_1 + x_{i_0} G_2$, which depends on i_0 variables and has the weight j_0. Of course, in this case the matrix of the new parameterization is obtain from the original one by the transformation, which is inverse to the elementary one.

Now for convenience denote

$$K_1 = N - n_1 + 1, \quad K_2 = w_2 - w_0 + 1. \tag{2.20}$$

Note that among all matrices, which can be obtained from the given $K_1 \times K_2$ matrix P with integer entries by a composition of a finite number of elementary transformations and their inverses, there exists a unique matrix, denoted by $\mathrm{Norm}(P)$, such that all its entries, except those lying on the first row and the last column, are equal to zero. To prove the existence of $\mathrm{Norm}(P)$ one can vanish the entries of the matrix P by a composition of elementary transformations and their inverses step by step, starting from the entry in the left-lower corner, going along the first column from the bottom to the top till the entry on the second row , then passing to the bottom of the second column, going along it from the bottom to the top till the entry on the second row and so on till the column before the last one. The uniqueness follows from the fact that if we put the entries of the matrix $\mathrm{Norm}(P)$ instead of the entries of P in the representation (2.16), then we obtain the (w_2, n_1)-representation of the Poincare function $M(t)$. This representation is unique according to Lemma 1 and the matrix $\mathrm{Norm}(P)$ is obviously uniquely recovered from it. Also it is not difficult to express all nontrivial entries of $\mathrm{Norm}(P)$ by the entries of P:

$$(\mathrm{Norm}(P))_{1j} = p_{1j} + \sum_{l=0}^{j-1} \sum_{k=1}^{K_1-1} \binom{k+l-1}{l} p_{k+1,j-l}, \qquad 1 \le j \le K_2 - 1, \tag{2.21a}$$

$$(\mathrm{Norm}(P))_{i,K_2} = p_{i,K_2} + \sum_{l=0}^{K_1-i+1} \sum_{k=1}^{K_2-1} \binom{K_2-k+l-1}{l} p_{i+l,k}, \qquad 2 \le i \le K_1, \tag{2.21b}$$

$$(\mathrm{Norm}(P))_{1,K_2} = p_{1,K_2} \tag{2.21c}$$

The last relation can be proved, for example, using the procedure of passing from P to $\mathrm{Norm}(P)$, described above, and the following well-known combinatorial identity:

$$\sum_{i=1}^{n} \binom{i+k-1}{k} = \binom{n+k}{k+1}.$$

If the matrix P has only nonnegative integer entries, then the matrix $\mathrm{Norm}(P)$ is obtained from P by a finite composition of elementary transformations (without using their inverses) and has also only nonnegative integer entries (which follows also from relations (2.21)). Moreover, if we put the entries of the matrix Norm(P) instead of the entries of P in the representation (2.16), then we obtain the nice (w_2, n_1)-representation of the Poincare function $M(t)$ of our classification problem. We also say that this nice representation *corresponds to the matrix* $\mathrm{Norm}(P)$. We can summarize all above in the following

Proposition 1 *If the classification problem admits a regular (w_2, n_1)-parameterization with parameterization matrix P, then it admits a regular (w_2, n_1)-parameterization with parameterization matrix* $\mathrm{Norm}(P)$ *and its Poincare series has the nice (w_2, n_1)-representation, which corresponds to the matrix* $\mathrm{Norm}(P)$.

The last proposition indicates that the nice representations of the Poincare series (if they exist) may be used in the definition of the intrinsic number of functional invariants of each number of variables and weight, on which the given classification problem depends. Suppose that the Poincare series $M(t)$ has the nice representation for some (w, l). So, the set

$$\mathrm{NS}\big(M(t)\big) \stackrel{def}{=} \{(w, l) : \text{the } (w, l)-\text{representation of } M(t) \text{ is nice}\} \tag{2.22}$$

is not empty. The natural question is what pair to choose from $\mathrm{NS}\big(M(t)\big)$? To answer this question we propose to introduce the order $\prec$ on the set of ordered pairs (w, l) in the following way: $(w, l) \prec (\bar{w}, \bar{l})$ if and only if $w < \bar{w}$ or $w = \bar{w}$, but $l > \bar{l}$. By Definition 3

$$\mathrm{NS}\big(M(t)\big) \subset \{(w, l) : w \geq w_0, l \leq N\}, \tag{2.23}$$

which implies immediately that the set $\mathrm{NS}\big(M(t)\big)$ contains the minimal element w.r.t. the introduced order $\prec$. This minimal element will be called *the characteristic pair of the classification problem.* Denote it by $(\bar{w}, \bar{l})$. Let $\mathcal{C}$ be the $(N - \bar{l} + 1) \times (\bar{w} - w_0 + 1)$ matrix such that the $(\bar{w}, \bar{l})$-representation of $M(t)$ corresponds to the matrix $\mathcal{C}$.

Definition 6 *The (ij)-entry of the matrix $\mathcal{C}$ is called the intrinsic number of the functional invariants of $i + \bar{l} - 1$ variables and the weight $j + w_0 - 1$ of the considered classification problem. The matrix $\mathcal{C}$ is called the characteristic matrix of the classification problem. Any regular $(\bar{w}, \bar{l})$-parameterization of the problem (if it exists) with the parameterization matrix $\mathcal{C}$ is called the characteristic regular parameterization.*

In general, it is better to have a parameterization, consisting of invariants, which have minimal possible weight and depend on maximal possible number of variables. Our definition of characteristic parameterization is in accordance with this goal. Actually the maximal weight of invariants, appearing in a characteristic regular parameterization is not greater than the maximal weight of invariants, appearing in any other regular parameterization. Besides the minimal number of variables in invariants of a characteristic regular parameterization is not less than the minimal number of variables in invariants of any other regular parameterization, having the same maximal weight of invariants as a characteristic one.

One can improve the formula (2.23) for the localization of the set $\mathrm{NS}\big(M(t)\big)$. Indeed let d be the degree of the rational function $M(t)$ at infinity. Namely, if $M(t) = \frac{Q_1(t)}{Q_2(t)}$, where $Q_1(t)$ and $Q_2(t)$ are polynomials, then $d = \deg Q_1(t) - \deg Q_2(t)$. Then

$$\mathrm{NS}\big(M(t)\big) \subset \{(w, l) : w \geq w_0, 1 \leq l \leq \min\,(w - d - 1, N)\}. \tag{2.24}$$

To prove (2.24) we actually have to prove that if the pair $(w, l) \in \mathrm{NS}\big(M(t)\big)$ then $l \leq w - d - 1$. Indeed, from (2.10) and (2.13) it follows that $d = \max\,(w - l_1 - 1, w_0 + \deg R - l_1 - 1)$. But first, since $l_1 > l$, we have $w - l_1 - 1 \leq w - l - 1$ and secondly, since $\deg R < w - w_0$, we have $w_0 + \deg R - l_1 - 1 < w - l - 1$. Therefore $d \leq w - l - 1$, Q.E.D.

From (2.24) it follows also that

$$\mathrm{NS}\big(M(t)\big) \subset \{(w, l) : w \geq \max\,(w_0, d + 2)\}. \tag{2.25}$$

The relations (2.24) and (2.25) may be useful in searching for the characteristic pair of the classification problem.

Conclusion about Example 1. The representation (2.18) of the Poincare series of the classification problem considered in Example 1 is its nice $(4, 1)$-representation. Let us show that

$(4,1)$ is the characteristic pair of the considered classification problem. Indeed, from (2.18) in the considered case $w_0 = 2$, $N = 2$, and $d = 2$. Hence from (2.25) it follows that

$$\mathrm{NS}\big(M(t)\big) \subset \{(w,l) : w \geq 4\}. \tag{2.26}$$

Further, using (2.24), one has $\mathrm{NS}\big(M(t)\big) \cap \{(w,l) : w = 4\} = \{(4,1)\}$, which together with (2.26) implies that $(4,1)$ is the minimal element of $\mathrm{NS}\big(M(t)\big)$. In other words, $(4,1)$ is the characteristic pair of our classification problem. Also, it implies that the characteristic matrix $\mathcal{C}$ of the problem is equal to the matrix P from (2.17) (note that in this case $\mathrm{Norm}(P) = P$). So, *the characteristic parameterization of set of germs of Riemannian metrics on an oriented two-dimensional Riemannian manifold consists of 1 functional invariant of 1 variables and the weight 2, 1 functional invariant of* 1 *variables and the weight 4, and 1 functional invariant of 2 variables and the weight 4.* □

Another useful property of the set $\mathrm{NS}\big(M(t)\big)$ can be formulated as follows:

Lemma 2 *Assume that the function $M(t)$ has the nice (w,l)-representation (2.10), the functions $R(t)$, $Q(t)$, and the number l_1 are as in (2.13), and $l_1 = l$ (or, equivalent, $q_l > 0$), then $(w-1, l-1) \notin \mathrm{NS}\big(M(t)\big)$.*

Proof. Let $S(t)$ be the polynomial such that $M(t) = \frac{S(t)}{(1-t)^{N+1}}$. Then, using the assumption $l = l_1$, it is easy to get

$$\deg S(t) = w + N - l. \tag{2.27}$$

Moreover, directly from (2.10) and (2.13) one can obtain that

$$\frac{d^{w+N-l} S}{d\, t^{w+N-l}} = (-1)^{N-l}(w+N-l)!\, q_l. \tag{2.28}$$

On the other hand, if the $(w-1, l-1)$-representation of $M(t)$ has the form

$$M(t) = \frac{t^{w_0} \sum_{i=0}^{w-w_0-2} \bar{r}_i t^i}{(1-t)^l} + t^{w-1} \sum_{j=l_2}^{N} \frac{\bar{q}_j}{(1-t)^{j+1}}, \quad \bar{q}_{l_2} \neq 0, \tag{2.29}$$

then $\deg S(t) = \max(w-1+N-l_2, w+N-l-1)$. Comparing this with (2.27) one gets easily that $l_2 = l-1$. But then by analogy with (2.28) (applied for $(w-1,l-1)$-representation instead of (w,l)-representation) one has

$$\frac{d^{w+N-l} S}{d\, t^{w+N-l}} = (-1)^{N-l+1}(w+N-l)!\, \bar{q}_{l-1}. \tag{2.30}$$

Comparing (2.28) and (2.30), we obtain that $\bar{q}_{l-1} = -q_l$. Hence $\bar{q}_{l-1} < 0$ and the $(w-1, l-1)$-representation (2.29) is not nice. □

As a direct consequence of Proposition 1, the previous lemma, and the relation (2.21c) one has the following

Corollary 1 *If the classification problem with the Poincare series $M(t)$ admits the regular (w,l)-parameterization with the parameterization matrix P such that the entry in the right-upper corner of P is positive (i.e., in the previous notations $p_{1,w-w_0+1} > 0$), then $(w-1, l-1) \notin \mathrm{NS}\big(M(t)\big)$.*

In the sequel we will show that all classification problems 1-4 listed in Introduction are regular. For each of this problems we will describe explicitly some its regular (w,l)-parameterization such that (w,l) is the characteristic pair of the problem and find its characteristic matrix, which also gives the way to obtain the characteristic parameterization.

Remark 3 In the classification problem of Example 1 and, as we will see later, in all classification problems 1-4, listed in Introduction, to any generic object one can assign the canonical coordinates. This allows us to construct functional invariants in the sense of Definition 1 from the invariants defined on the ambient manifold (in the same manner, as in Example 1 we constructed the functional invariant $\mathfrak{K}$ from the Gaussian curvature). Sometimes (as in the case of (1,3) control-affine systems and in the case of Riemannian metrics on non-oriented two dimensional manifold) the canonical coordinates (even for generic objects) are defined up to some discrete group of transformations (in the mentioned cases a group of reflections). Such classification problems can be treated in a similar way after slight modifications of definitions of the functional invariants, the parametrization, the weight and the regular parametrization.

Indeed, a discrete group Γ on $\mathbb{R}^l$ induces the action $\mathfrak{G}$ on $C_0^\infty(\mathbb{R}^l,\mathbb{R})$ in the obvious way. By functional quasi-invariant we mean the mapping from the generic subset $\widetilde{\mathcal{O}}$ of the set of considered objects to the set of orbits w.r.t. the action $\mathfrak{G}$ on $C_0^\infty(\mathbb{R}^l,\mathbb{R})$ such that this mapping is invariant w.r.t. the action of the group Diff_{q_0} on $\tilde{\mathcal{O}}$. The notions of the weight of the functional quasi-invariants, the parametrization and the regular parametrization by functional quasi-invariants can be defined in the natural way. Besides, in parametrizations we can admit invariants taking their values in some discrete set (or, shortly, discrete invariants) and define the regular parameterization by functional (quasi-)invariants and discrete invariants in the natural way. Permission of the functional quasi-invariant and the discrete invariants does not affect on the formula (2.16) for the Poincare series in terms of the parameterization matrix, containing the information about the number of functional (quasi-)invariants of the given number of variables and weight in the parameterization. So, also the characteristic pair, the characteristic matrix, and the characteristic regular parametrization can be defined in this case too. In the case of Riemannian metrics on the non-oriented manifolds the canonical coordinates are defined up to the reflection $(x_1,x_2)\mapsto(x_1,-x_2)$, the characteristic pair and the characteristic matrix are as in the oriented case, the characteristic parametrization is defined by formula (2.4), where we take into account both canonical coordinates (x_1,x_2) and $(x_1,-x_2)$.

Further, according to Proposition 3.2 of [1] (see also Remark 1 and the paragraph before it in Introduction), the state-feedback classification problem for $(1,3)$ control-affine system admits regular $(2,2)$-parameterization by one discrete invariant taking values in $\{-1,1\}$ and the functional quasi-invariants with the following 2×2 parameterization matrix $P=\begin{pmatrix}2&0\\0&1\end{pmatrix}$. Besides, $w_0=1$ and $N=3$. By the inverse to the elementary transformation one can transform P to the matrix $\widetilde{P}=\begin{pmatrix}1&0\\1&0\end{pmatrix}$. The Poincare series $M(t)$ of the problem satisfies

$$M(t)=t\left(\frac{1}{(1-t)^3}+\frac{1}{(1-t)^4}\right).$$

It is not difficult to see that by erasing the last column of the matrix $\tilde{P}$ one obtains the characteristic matrix $\mathcal{C}=(1,1)^T$ of the considered classification problem and the characteristic pair is equal to $(1,2)$. So, *the characteristic regular parameterization of the state-feedback classification problem for* $(1,3)$ *control-affine systemconsists of one functional quasi-invariant of 3 variables* *and the weight* 1*, one functional quasi-invariant of 2 variables and the weight* 1*, and the discrete invariant from the set* $\{-1, 1\}$*.* This parameterization is obtained from the original one by a rearrangement of the invariants, which corresponds to the inverse to elementary transformation, transforming the matrix P to the matrix $\widetilde{P}$ (such rearrangements were described in the paragraph after the formula (2.19)).□

3 Classification of $(1, n)$ control-affine systems for $n \geq 4$

For $r = 1$ the system (1.1) has the following form

$$\dot{q} = f_0(q) + u\, f_1(q), \quad q \in M, \quad u \in \mathbb{R}. \tag{3.1}$$

Our genericity assumptions are

$$\dim \operatorname{span}(f_0, f_1, [f_1, f_0], \dots (\operatorname{ad} f_1)^{n-2} f_0) = n, \tag{3.2}$$

$$\dim \operatorname{span}(f_1, \big[f_0, [f_0, f_1]\big], [f_1, f_0], \dots (\operatorname{ad} f_1)^{n-2} f_0) = n, \tag{3.3}$$

and for $n \geq 5$ also

$$\dim \operatorname{span}(f_0, f_1, \big[f_0, [f_0, f_1]\big], [f_1, f_0], \dots (\operatorname{ad} f_1)^{n-3} f_0) = n. \tag{3.4}$$

The group of feedback transformations

$$u = \beta(q)\tilde{u} + \alpha(q), \beta(q) \neq 0 \tag{3.5}$$

acts naturally on the set of pairs of vector fields (f_0, f_1). The orbit w.r.t. this action is

$$\mathcal{O}_{(f_0, f_1)} = \{(f_0 + \alpha f_1, \beta f_1) : \alpha, \beta : M \mapsto \mathbb{R} \text{ are functions}, \beta \neq 0\}. \tag{3.6}$$

The first observation is given by the following

Proposition 2 *If the pair* (f_0, f_1) *satisfies conditions (3.2) and (3.3), then there exists a unique pair* $(F_0, F_1) \in \mathcal{O}_{(f_0, f_1)}$ *such that*

$$\big[F_0, [F_0, F_1]\big] = F_0 + I_1 F_1 + I_2 [F_1, F_0] + \sum_{k=3}^{n-2} I_k (\operatorname{ad} F_1)^k F_0 \tag{3.7}$$

Proof. By assumptions (3.2) the vector fields $f_0, f_1, [f_1, f_0], \dots (\operatorname{ad} f_1)^{n-2} f_0$ constitute the frame on M. Therefore there are functions E, N, $J_1, \dots, J_{n-2}$ such that

$$\big[f_0, [f_0, f_1]\big] = E f_0 + J_1 f_1 + J_2 [f_1, f_0] + Z\big[f_1, [f_1, f_0]\big] + \sum_{k=3}^{n-2} J_k (\operatorname{ad} f_1)^k f_0 \tag{3.8}$$

Take some pair $(\tilde{f}_0, \tilde{f}_1) \in \mathcal{O}_{(f_0, f_1)}$,

$$\tilde{f}_0 = f_0 + \alpha f_1, \quad \tilde{f}_1 = \beta f_1. \tag{3.9}$$

Suppose that

$$[\tilde{f}_0, [\tilde{f}_0, \tilde{f}_1]] = \widetilde{E} \tilde{f}_0 + \widetilde{J}_1 \tilde{f}_1 + \widetilde{J}_2 [\tilde{f}_1, \tilde{f}_0] + \widetilde{Z}\big[\tilde{f}_1, [\tilde{f}_1, \tilde{f}_0]\big] + \sum_{k=3}^{n-2} J_k (\operatorname{ad} \tilde{f}_1)^k \tilde{f}_0 \tag{3.10}$$

First note that

$$\widetilde{E} = \beta E. \tag{3.11}$$

It follows immediately from (3.9) and the following relations

$$[\tilde{f}_0, [\tilde{f}_0, \tilde{f}_1]] \equiv \beta[f_0, [f_0, f_1]] - \alpha\beta[f_1, [f_1, f_0]] \bmod \operatorname{span}(f_1, [f_0, f_1]), \tag{3.12}$$

$$(\operatorname{ad} f_1)^k f_0 \in \operatorname{span}(\tilde{f}_1, \ldots, (\operatorname{ad}\tilde{f}_1)^k \tilde{f}_0), \quad k \in \mathbb{N}. \tag{3.13}$$

From assumption (3.3) it follows that $E \neq 0$. Therefore, taking $\beta = \frac{1}{E}$, we make

$$\widetilde{E} = 1. \tag{3.14}$$

Let us denote by $\overline{\mathcal{O}}_{(f_0,f_1)}$ the set of all pairs $(\tilde{f}_0, \tilde{f}_1)$, satisfying (3.14). We can assume from the beginning that the original pair (f_0, f_1) belongs to $\overline{\mathcal{O}}_{(f_0,f_1)}$, i.e. $E = 1$ (we make this assumption just in order to avoid extra notations). If $(\tilde{f}_0, \tilde{f}_1) \in \overline{\mathcal{O}}_{(f_0,f_1)}$, then also $\tilde{E} = 1$. Hence $\beta = 1$ or equivalently $f_1 = \tilde{f}_1$. In other words, condition (3.14) normalizes the vector field f_1 or the direction defining the straight line in the set of admissible velocities of the system (3.1) at any point.

Further, from (3.9), taking into account that $\beta = 1$, it follows easily that

$$(\operatorname{ad} f_1)^k f_0 \equiv (\operatorname{ad}\tilde{f}_1)^k \tilde{f}_0 \bmod \operatorname{span}(f_1), \quad k \in \mathbb{N}.$$

This and relation (3.12) imply that

$$\widetilde{Z} = Z - \alpha. \tag{3.15}$$

Setting $\alpha = Z$, we make $\tilde{Z} = 0$, which normalizes the drift $\tilde{f}_0$. So, we have proved that there is a unique $(\tilde{f}_0, \tilde{f}_1) \in \mathcal{O}_{(f_0,f_1)}$ such that $\tilde{E} = 1$ and $\tilde{Z} = 0$, which completes the proof of the proposition. □

Remark 4 The mappings $I_1, \ldots, I_{n-2}$ from M to $\mathbb{R}$, defined by identity (3.7), are state-feedback invariants of the control system (3.1). □

The vector field F_0 and the pair of vector fields (F_0, F_1) from Proposition 2 are called *the canonical drift* and *the canonical pair* of the system (3.1) respectively.

Remark 5 Actually, in the case $n = 4$, the vector $F_0(q)$ is the velocity of the unique abnormal extremal starting at q of the time optimal problem defined by system (3.1). □

Now fix some point $q_0 \in M$. Denote by e^{tf} the flow, generated by the vector field f and $q \circ e^{tf}$ the image of the point q w.r.t. this flow. Let $\Phi_n : \mathbb{R}^n \mapsto M$ be the following mapping

$$\Phi_4(x_1, x_2, x_3, x_4) = q_0 \circ e^{x_4 [F_1,[F_1,F_0]]} \circ e^{x_3[F_1,F_0]} \circ e^{x_2 F_1} \circ e^{x_1 F_0}, \tag{3.16}$$

$$\begin{aligned} \Phi_n(x_1, \ldots, x_n) = \; & q_0 \circ e^{x_n (\operatorname{ad} F_1)^{n-3} F_0} \circ \ldots \circ e^{x_5 [F_1,[F_1,F_0]]} \circ \\ & \circ e^{x_4 [F_0,[F_0,F_1]]} \circ e^{x_3[F_1,F_0]} \circ e^{x_2 F_0} \circ e^{x_1 F_1}, \quad n \geq 5 \end{aligned} \tag{3.17}$$

From assumption (3.2) in the case $n = 4$ or assumption (3.4) in the case $n \geq 5$ it follows that $\Phi_n'(0)$ is bijective. Hence Φ_n^{-1} defines *the canonical coordinates in a neighborhood of* q_0 (or shortly *the canonical coordinates at* q_0). Denote

$$\mathcal{I}_k = I_k \circ \Phi_n, \quad k = 1, \ldots n-2. \tag{3.18}$$

Assigning to any generic germ at q_0 of control-affine systems (3.1) the function $\mathcal{I}_k$ we obtain the functional invariant of n variables of this set of objects in the sense of Definition 1 for any $1 \leq k \leq n-2$.

Now let us consider the cases $n = 4$ and $n \geq 5$ separately:

a) The case $n = 4$**.** By (3.16) and (3.7), in the canonical coordinates the vector fields F_0 and F_1 have the following form:

$$F_0 = \frac{\partial}{\partial x_1}, \quad F_1 = \sum_{k=1}^{4} a_k \frac{\partial}{\partial x_k}, \tag{3.19}$$

where the components of F_1 satisfy the following second order linear ordinary differential equations w.r.t. the variable x_1

$$\frac{\partial^2 a_k}{\partial x_1^2} + \mathcal{I}_2 \frac{\partial a_k}{\partial x_1} - \mathcal{I}_1 a_k - \delta_{1,k} = 0 \quad k = 1, 2, 3, 4, \tag{3.20}$$

with the following restrictions on the initial conditions for any $k = 1, 2, 3, 4$

$$a_k(0, x_2, x_3, x_4) \equiv \delta_{2k}, \tag{3.21a}$$

$$\frac{\partial a_k}{\partial x_1}(0, 0, x_3, x_4) \equiv -\delta_{3k}, \tag{3.21b}$$

$$\frac{\partial^2 a_k}{\partial x_1 \partial x_2}(0, 0, 0, x_4) \equiv -\delta_{4k}, \tag{3.21c}$$

where δ_{ij} is the Kronecker symbol. Let for any $k = 1, 2, 3, 4$

$$\beta_k(x_2, x_3, x_4) \stackrel{def}{=} \frac{\partial^3 a_k}{\partial x_1 \partial x_2^2}(0, x_2, x_3, x_4), \tag{3.22a}$$

$$\psi_k(x_3, x_4) \stackrel{def}{=} \frac{\partial^3 a_k}{\partial x_1 \partial x_2 \partial x_3}(0, 0, x_3, x_4). \tag{3.22b}$$

So, with any germ at q_0 of a four-dimensional affine system (3.1), satisfying genericity assumptions (3.2) and (3.3), one can associate the ordered tuple

$$(\mathcal{I}_1, \mathcal{I}_2, \beta_1, \beta_2, \beta_3, \beta_4, \psi_1, \psi_2, \psi_3, \psi_4), \tag{3.23}$$

of state-feedback functional invariants, consisting of two germs $\mathcal{I}_1$ and $\mathcal{I}_2$ of functions of four variables at 0, four germs $\beta_1, \beta_2, \beta_3, \beta_4$ of functions of three variables at 0, and four germs $\psi_1, \psi_2, \psi_3, \psi_4$ of functions of three variables at 0. We call it *the tuple of the primary invariants of the* $(1, 4)$ *control-affine system* (3.1) *at the point* q_0. Note that by (3.22) the functional invariants β_k and ψ_k have the weight 3 for any $1 \leq k \leq 4$, while by (3.20) and (3.21) the functional invariants $\mathcal{I}_1$ and $\mathcal{I}_2$ have the weight 2.

Further, fixing β_k and ψ_k and using (3.21b) and (3.21c), one can find $\frac{\partial a_k}{\partial x_1}(0, x_2, x_3, x_4)$ for any $1 \leq k \leq 4$ by the appropriate integrations (see (3.31) below). If in turn one fixes also $\mathcal{I}_1$ and $\mathcal{I}_2$, then from the knowledge of $\frac{\partial a_k}{\partial x_1}(0, x_2, x_3, x_4)$, condition (3.21a), and differential equation (3.20) we can recover the functions $a_k(x_1, x_2, x_3, x_4)$ and therefore our control-affine system itself, just using the standard existence and uniqueness results from the theory of ordinary differential equations. We summarize all above in the following:

Theorem 1 *Given two arbitrary germs $\mathcal{I}_1$ and $\mathcal{I}_2$ of functions of four variables at* 0*, four arbitrary germs $\beta_1, \beta_2, \beta_3, \beta_4$ of functions of three variables at* 0*, and four arbitrary germs $\psi_1, \psi_2, \psi_3, \psi_4$ of functions of two variables at* 0 *there exists a unique, up to state-feedback transformation of the type (1.3), four-dimensional control-affine system with scalar input, satisfying genericity assumptions (3.2) and (3.3), such that the tuple $(\mathcal{I}_1, \mathcal{I}_2, \beta_1, \beta_2, \beta_3, \beta_4, \psi_1, \psi_2, \psi_3, \psi_4)$ is its tuple of the primary invariants at the given point q_0. In other words, the tuples of the primary invariants give the regular* $(3,2)$*-parameterization of the considered classification problem with the following* 3×2 *parameterization matrix* P*:*

$$P = \begin{pmatrix} 0 & 4 \\ 0 & 4 \\ 2 & 0 \end{pmatrix}. \tag{3.24}$$

The Poincare series $M(t)$ of the considered classification problem satisfies

$$M(t) = \frac{2t^2}{(1-t)^5} + t^3 \left(\frac{4}{(1-t)^4} + \frac{4}{(1-t)^3} \right). \tag{3.25}$$

It turns out that $(3,2)$ is the characteristic pair of the considered classification problem. Indeed, let as before w_0 be the order of zero of $M(t)$ at $t = 0$, N is the order of pole of $(1-t)M(t)$ at $t = 1$, and d is the degree of $M(t)$ (at infinity). Then from (3.25) it follows that $w_0 = 2$, $N = 4$, and $d = 0$. Hence from (2.23) (or (2.25)) it follows that

$$\mathrm{NS}\big(M(t)\big) \subset \{(w,l) : w \geq 2\}. \tag{3.26}$$

Further, using (2.24), one has

$$\mathrm{NS}\big(M(t)\big) \cap \{(w,l) : w = 2\} \subset \{(2,1)\}, \quad \mathrm{NS}\big(M(t)\big) \cap \{(w,l) : w = 3\} \subset \{(3,1),(3,2)\}. \tag{3.27}$$

But by the previous theorem our classification problem admits $(3,2)$-parameterization such that its parameterization matrix has a positive entry in the right-upper corner. Therefore from Corollary 1 it follows that $(2,1) \notin \mathrm{NS}\big(M(t)\big)$. Since $(3,2) \prec (3,1)$ we can conclude from (3.26) and (3.27) that $(3,2)$ is the minimal element of $\mathrm{NS}\big(M(t)\big)$. In other words, $(3,2)$ is the characteristic pair of our classification problem. Also, it implies that the characteristic matrix $\mathcal{C}$ of the problem is equal to $\mathrm{Norm}(P)$, which can be found easily by the series of elementary transformations. Namely,

$$\mathcal{C} = \mathrm{Norm}(P) = \begin{pmatrix} 2 & 4 \\ 0 & 6 \\ 0 & 2 \end{pmatrix}. \tag{3.28}$$

Conclusion 1 *The characteristic parameterization of* $(1,4)$ *control-affine systems, up to state-feedback transformations, consists of* 2 *functional invariants of* 4 *variables and the weight* 3*,* 6 *functional invariants of* 3 *variables and the weight* 3*,* 2 *functional invariants of* 2 *variables and the weight* 2*, and* 4 *functional invariants of* 2 *variables and the weight* 3*.*

In order to obtain a characteristic parameterization from the parameterization by the tuple of the primary invariants one can implement some series of rearrangement of the primary invariants according to the series of elementary transformations from the matrix P to $\mathrm{Norm}(P)$, as was described in the previous section (see, for example, formula (2.19) and the paragraph after it).

We finish the treatment of the case of $(1,4)$-affine control systems by writing the local normal form of such system, up to state-feedback transformation, in terms of the tuple of their primary invariants: Let N be the solution of the following non-homogeneous second order linear ordinary differential equation w.r.t. the variable x_1 with prescribed initial values

$$\begin{cases} \dfrac{\partial^2 N}{\partial x_1^2} + \mathcal{I}_2 \dfrac{\partial N}{\partial x_1} - \mathcal{I}_1 N - 1 = 0; \\ N(0, x_2, x_3, x_4) \equiv 0, \quad \dfrac{\partial N}{\partial x_1}(x_1, x_2, x_3, x_4)\Big|_{x_1=0} \equiv 0, \end{cases} \tag{3.29}$$

and the functions ρ_1, ρ_2 be the solution of the following homogeneous second order linear ordinary differential equations w.r.t. the variable x_1 with prescribed initial values

$$\begin{cases} \dfrac{\partial^2 \rho_i}{\partial x_1^2} + \mathcal{I}_2 \dfrac{\partial \rho_i}{\partial x_1} - \mathcal{I}_1 \rho_i = 0, \quad i = 1, 2, \\ \begin{pmatrix} \rho_1(0, x_2, x_3, x_4) & \rho_1(0, x_2, x_3, x_4) \\ \frac{\partial}{\partial x_1}\rho_1(0, x_2, x_3, x_4) & \frac{\partial}{\partial x_1}\rho_2(0, x_2, x_3, x_4) \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{cases}. \tag{3.30}$$

Let also

$$B_k(x_2, x_3, x_4) = -\delta_{3k} + x_2\left(-\delta_{4k} + \int_0^{x_3} \psi_k(y, x_4)\, dy\right) + \int_0^{x_2} (x_2 - y)\beta_k(y, x_3, x_4)\, dy \tag{3.31}$$

for $1 \leq k \leq 4$ (actually $B_k(x_2, x_3, x_4) = \frac{\partial a_k}{\partial x_1}(0, x_2, x_3, x_4)$, where the functions a_k are as in (3.19)). Then a four-dimensional control-affine system (3.1) with the tuple of the primary invariants $(\mathcal{I}_1, \mathcal{I}_2, \beta_1, \beta_2, \beta_3, \beta_4, \psi_1, \psi_2, \psi_3, \psi_4)$ at the point q_0 is state-feedback equivalent to the following system:

$$\begin{cases} \dot{x}_1 = 1 + (N + B_1\rho_2)u \\ \dot{x}_2 = (\rho_1 + B_2\rho_2)u \qquad , \qquad u \in \mathbb{R}. \\ \dot{x}_i = B_i\rho_2 u, \quad i = 3, 4 \end{cases} \tag{3.32}$$

b) The case $n \geq 5$. By (3.17) and (3.7), in the canonical coordinates the vector fields F_0 and F_1 have the following form:

$$F_0 = \sum_{k=1}^{n} a_k \frac{\partial}{\partial x_k}, \quad F_1 = \frac{\partial}{\partial x_1}, \tag{3.33}$$

where the components a_k of F_0 satisfy the following system of partial differential equations

$$\mathcal{I}_{n-2}\frac{\partial^{n-2} a_k}{\partial x_1^{n-2}} + \sum_{j=3}^{n-3} \mathcal{I}_j \frac{\partial^j a_k}{\partial x_1^j} + \sum_{l=1}^{n}\left(a_l \frac{\partial^2 a_k}{\partial x_1^2} - \frac{\partial a_l}{\partial x_1}\frac{\partial a_k}{\partial x_l}\right) + \mathcal{I}_2 \frac{\partial a_k}{\partial x_1} + a_k + \mathcal{I}_1 \delta_{1,k} = 0, \quad k = 1, \dots, n, \tag{3.34}$$

with the following restrictions on the boundary conditions for any $1 \leq k \leq n$

$$a_k(0, x_2, \dots, x_n) \equiv \delta_{2k}, \tag{3.35a}$$

$$\frac{\partial a_k}{\partial x_1}(0, 0, x_3 \dots, x_n) \equiv \delta_{3k}, \tag{3.35b}$$

$$\frac{\partial^2 a_k}{\partial x_1 \partial x_2}(0, 0, 0, x_4, \dots, x_n) \equiv -\delta_{4k}, \tag{3.35c}$$

$$\frac{\partial^j a_k}{\partial x_1^j}(0, \dots, 0, x_{j+1}, \dots, x_n) \equiv \delta_{j+3,k}, \; 2 \leq j \leq n-3, \tag{3.35d}$$

where δ_{ij} is the Kronecker symbol. Note also that the genericity assumption (3.4) implies that

$$\mathcal{I}_{n-2} \neq 0. \tag{3.36}$$

Let us introduce the following functions for any $1 \le k \le n$:

$$\beta_k(x_2, \dots, x_n) \stackrel{def}{=} \frac{\partial^3 a_k}{\partial x_1 \partial x_2^2}(0, x_2, \dots, x_n), \tag{3.37a}$$

$$\psi_k(x_3, \dots, x_n) \stackrel{def}{=} \frac{\partial^3 a_k}{\partial x_1 \partial x_2 \partial x_3}(0, 0, x_3, \dots, x_n), \tag{3.37b}$$

$$\phi_{kjl}(x_l, \dots, x_n) \stackrel{def}{=} \frac{\partial^{j+1} a_k}{\partial x_1^j \partial x_l} a_k(0, \dots, 0, x_l, \dots x_n), \quad 2 \le j \le n-3, \ 2 \le l \le j+2. \tag{3.37c}$$

So, with any germ at q_0 of an n-dimensional affine system (3.1), satisfying genericity assumptions (3.2) and (3.3), one can associate the ordered tuple

$$\Big(\{\mathcal{I}_s(x_1, \dots, x_n)\}_{s=1}^{n-2}, \{\beta_k(x_2, \dots, x_n)\}_{k=1}^{n}, \{\psi_k(x_3, \dots, x_n)\}_{k=1}^{n}, \\ \{\phi_{kjl}(x_l, \dots, x_n) : 1 \le k \le n, 2 \le j \le n-3, 2 \le l \le j+2\} \Big) \tag{3.38}$$

of state-feedback invariants. of functions of $n-1$ variables at 0. We call it *the tuple of the primary invariants of the* $(1, n)$*-affine control system* (3.1) *with* $n > 4$ *at the point* q_0. Note that by (3.22) for any $1 \le k \le n$ the functional invariants β_k and ψ_k have the weight 3 , the functional invariants ϕ_{kjl} have the weight $j+1$, while by (3.20) and (3.21) for any $1 \le s \le n-2$ the functional invariants $\mathcal{I}_s$ have the weight $n-2$.

Further, fixing β_k and ψ_k and using (3.35b) and (3.35c), one can find $\frac{\partial a_k}{\partial x_1}(0, x_2, \dots, x_n)$ for any $1 \le k \le n$ by the appropriate integrations. Similarly, fixing $\{\phi_{kjl}\}_{l=2}^{j+2}$ for given j, $2 \le j \le n-3$, and using (3.35d), one can find $\frac{\partial^j a_k}{\partial x_1^j}(0, x_2, \dots, x_n)$ for any $1 \le k \le n$ by the appropriate integrations. Finally, if we suppose that all functions β_k, ψ_k, and ϕ_{kjl} are real analytic and fix also real analytic $\{\mathcal{I}_s\}_{s=1}^{n-2}$, then from the knowledge of $\frac{\partial^j a_k}{\partial x_1^j}(0, x_2, \dots, x_n)$ for all $1 \le j \le n-3$, condition (3.35a), and differential equation (3.34) we can recover the functions $a_k(x_1, \dots, x_n)$ and therefore our affine control system itself, just using the classical Cauchy - Kowalewsky theorem for system (3.34). We summarize all above in the following:

Theorem 2 *If* $n \ge 5$*, then given an arbitrary tuple* (3.38) *of real analytic functions* $n-2$ *germs* $\{\mathcal{I}_j\}_{j=1}^{n-2}$ *of real analytic functions there exists a unique, up to state-feedback real analytic transformation of the type (1.3),* n*-dimensional real analytic control-affine system with scalar input, satisfying genericity assumptions (3.2),(3.3), and (3.4),such that the tuple* (3.38) *is its tuple of the primary invariants at the given point* q_0*. In other words, the tuples of the primary invariants give the regular* $(n-2, 2)$*-parameterization of the considered classification problem in the real analytic category with the* $(n-1) \times (n-4)$ *parameterization matrix* P *such that for* $n = 5$

$$P = (5, 10, 10, 3)^T \tag{3.39}$$

and for $n > 5$

$$P = \begin{pmatrix} 0 & \dots\dots\dots & 0 & n \\ \vdots & & \iddots\ \iddots & \vdots \\ \vdots & \iddots\ \iddots & & \vdots \\ \vdots\ \iddots\ \iddots & & & \vdots \\ 0 & n & \dots\dots\dots & n \\ n & n & \dots\dots\dots & n \\ 2n & n & \dots\dots\dots & n \\ 2n & n & \dots\dots\dots & n \\ 0 & \dots\dots\dots & 0 & n-2 \end{pmatrix} \tag{3.40}$$

(in the matrix P all entries in the triangle with vertices in the $(1,1)$-entry, the $(1,n-5)$-entry, and the $(n-5,1)$-entry are equal to 0, while all entries in the triangle with vertices in the $(1,n-4)$-entry, the $(n-4,1)$-entry, and the $(n-4,n-4)$-entry are equal to n). The Poincare series $M(t)$ of the considered classification problem satisfies

$$\begin{aligned} M(t) = nt^3 &\left(\frac{2}{(1-t)^n} + \frac{2}{(1-t)^{n-1}} + \frac{1}{(1-t)^{n-2}}\right) \\ &+ n\sum_{i=4}^{n-2} t^i \sum_{j=n-i}^{n-1} \frac{1}{(1-t)^{j+1}} + \frac{(n-2)t^{n-2}}{(1-t)^{n+1}}. \end{aligned} \tag{3.41}$$

It turns out that $(n-2,2)$ is the characteristic pair of the considered classification problem. Indeed, let as before w_0 be the order of zero of $M(t)$ at $t=0$, N is the order of pole of $(1-t)M(t)$ at $t=1$, and d is the degree of $M(t)$ (at infinity). Then from (3.25) it follows that $w_0 = 3$, $N = n$, and $d = n-5$. Hence from (2.25) it follows that

$$\mathrm{NS}\big(M(t)\big) \subset \{(w,l) : w \geq n-3\}. \tag{3.42}$$

Further, using (2.24), one has

$$\begin{aligned} &\mathrm{NS}\big(M(t)\big) \cap \{(w,l) : w = n-3\} \subset \{(n-3,1)\}, \\ &\mathrm{NS}\big(M(t)\big) \cap \{(w,l) : w = n-2\} \subset \{(n-2,1),(n-2,2)\}. \end{aligned} \tag{3.43}$$

But by the previous theorem our classification problem admits $(n-2,2)$-parameterization such that its parameterization metric has a positive entry in the right-upper corner. Therefore from Corollary 1 it follows that $(n-3,1) \notin \mathrm{NS}\big(M(t)\big)$. Since $(n-2,2) \prec (n-2,1)$ we can conclude from (3.42) and (3.43) that $(n-2,2)$ is the minimal element of $\mathrm{NS}\big(M(t)\big)$. In other words, $(n-2,2)$ is the characteristic pair of our classification problem. Also, it implies that the characteristic matrix $\mathcal{C}$ of the problem is equal to $\mathrm{Norm}(P)$, where P is as in Theorem 2. If $n=5$ then obviously $\mathcal{C} = \mathrm{Norm}(P) = P$. For $n > 5$ one can calculate all nontrivial entries of $\mathrm{Norm}(P)$, using identities (2.21). It gives all nontrivial entries of the characteristic matrix $\mathcal{C}$:

$$\begin{aligned}
&\mathcal{C}_{n-1,n-4} = n-2,\\
&\mathcal{C}_{n-2,n-4} = n(n-3),\\
&\mathcal{C}_{i,n-4} = n\Big(1+\sum_{l=1}^{n-3-i}\tbinom{i+2l-3}{l}+\tbinom{2n-9-i}{n-3-i}+2\tbinom{2n-8-i}{n-2-i}+\tbinom{2n-7-i}{n-7-i}\Big),\quad 2\le i\le n-3,\\
&\mathcal{C}_{1,n-4} = n,\\
&\mathcal{C}_{1j} = n\Big(\tbinom{n-4+j}{j-1}+2\tbinom{n-5+j}{j-1}+2\tbinom{n-6+j}{j-1}+\tbinom{n-7+j}{j-1}-1-\\
&-\sum_{l=1}^{j-1}\tbinom{n+2l-7-j}{l}\Big),\quad 1\le j\le n-5.
\end{aligned}\tag{3.44}$$

Recall that $\mathcal{C}_{ij}$ is the intrinsic number of the functional invariants of $i+1$ variables and the weight $j+2$. In order to obtain a characteristic parameterization from the parameterization by the tuple of the primary invariants one can implement some series of rearrangement of the primary invariants according to the series of elementary transformations from the matrix P to $\mathrm{Norm}(P)$, as was described in the previous section (see, for example, the formula (2.19) and the paragraph after it).

Remark 6 Two control systems $\dot{y}=\mathcal{F}(y,v)$ and $\dot{\tilde{y}}=\widetilde{\mathcal{F}}(\tilde{y},\tilde{v})$, with m-dimensional state-space S and one dimensional control space V are called *micro-locally state-feedback equivalent at the point* $(y_0,v_0)\in S\times V$, if there exist the state-feedback transformation

$$\begin{cases}\tilde{y}=\Phi_1(y)\\ \tilde{v}=\Phi_2(y,v)\\ y_0=\Phi_1(y_0),\ v_0=\Phi_2(y_0,v_0)\end{cases}$$

such that in the neighborhood of (y_0,v_0) in $S\times V$ the following identity holds:

$$\widetilde{\mathcal{F}}\big(\Phi_1(y),\Phi_2(y,v)\big)=d\Phi_1\mathcal{F}(y,v).$$

The affine $(1,m+1)$ control system (1.5) will be called *the affine extension* of the control system (1.4). It is not difficult to show that two control systems with scalar input are micro-locally state-feedback equivalent at some point $(y_0,v_0)\in S\times V$ if and only if their affine extensions are locally equivalent w.r.t. the state-feedback transformations of the type (1.3) at the same point. Note also that the affine extensions of generic m-dimensional control systems with scalar input are generic in the set of all $(1,m+1)$-affine systems. Using this fact and Theorems 1, 2 one obtains the micro-local parameterization of non-affine m-dimension control systems with scalar inputs and $m\ge 3$ by the tuples of the primary invariants of their affine extensions (in C^∞ category for $m=3$ and C^ω category for $m\ge 4$). Obviously, the Poincare series, the characteristic pair and the characteristic matrix of the micro-local state-feedback classification problem for m-dimensional control systems with scalar input are exactly the same as in the case of the state-feedback classification problem of $(1,m+1)$ control-affine systems. Besides, since in our method of normalization of $(1,n)$ control-affine system with $n\ge 5$ we rectify the vector field f_1, then in the case $m\ge 4$ a generic m-dimensional real analytic control system with the prescribed tuple (3.38) of the primary invariants of its affine extension is micro-locally state-feedback equivalent in the real analytic category to the system

$$\dot{y}_s=f_s(y_1,\dots,y_m,v),\quad 1\le s\le m,$$

such that $f_s(x_2, x_3, \dots x_m, x_1) = a_{s+1}(x_1, \dots x_{m+1})$, where the tuple $\{a_k(x_1, \dots, x_m)\}_{k=1}^{m+1}$ is the solution of the system of partial differential equations (3.34) with boundary conditions, which can be expressed by the primary invariants β_k, ψ_k, and ϕ_{kjl}, using (3.37).

4 Reduction of control-affine systems with two-dimensional input to the scalar input case in dimensions four and five

For $r = 2$ the system (1.1) has the following form

$$\dot{q} = f_0(q) + u_1\, f_1(q) + u_2\, f_2(q), \quad q \in M, \quad u_1,\, u_2 \in \mathbb{R}. \tag{4.1}$$

Our aim is to assign to the system (4.1) in a canonical way an affine subsystem with scalar input[1]. It turns out that in the case $n = 4$ the original system can be recovered from it uniquely up to a feedback transformation, while in the case $n = 5$ such unique recovering is possible after introducing an additional invariant function of n variables (which is natural in view of the estimates for the number of functional parameters, given in the Introduction).

4.1 Preliminaries. Let us look on (4.1), as on the time optimal control problem and find its extremals. First we introduce some notations. Let T^*M be the cotangent bundle of M with canonical symplectic form σ. Denote by h_i, $0 \le i \le 2$, the following functions on T^*M:

$$h_i(\lambda) = p \cdot f_i(q),\ \lambda = (p, q),\ q \in M,\ p \in T_q^*M. \tag{4.2}$$

For a given function $G : T^*M \mapsto \mathbb{R}$ denote by $\vec{G}$ the corresponding Hamiltonian vector field defined by the relation $\sigma(\vec{G}, \cdot) = -d\,G(\cdot)$. For a given vector distribution D on M (i.e., a subbundle of the tangent bundle), define the lth power D^l by the recursive relation

$$D^l = D^{l-1} + [D, D^{l-1}], \quad D^1 = D,$$

and denote by $(D^l)^\perp \subset T^*M$ the annihilator of D^l, namely

$$(D^l)^\perp = \{(p, q) \in T^*M : p \cdot v = 0\ \forall v \in D^l(q)\}.$$

In the introduced notations the Hamiltonian of Pontryagin Maximum Principle for the time optimal problem (4.1) can be written as follows:

$$H(\lambda, u_1, u_2) = h_0(\lambda) + u_1 h_1(\lambda) + u_2 h_2(\lambda), \quad \lambda \in T^*M,\ u_1, u_2 \in \mathbb{R}. \tag{4.3}$$

Let $\gamma(\cdot)$ be an extremal of (4.1) with extremal control functions $\bar{u}_1(t)$ and $\bar{u}_2(t)$. Then

$$\dot{\gamma}(t) = \vec{h}_0\big(\gamma(t)\big) + \bar{u}_1(t)\vec{h}_1\big(\gamma(t)\big) + \bar{u}_2(t)\vec{h}_2\big(\gamma(t)\big) \tag{4.4}$$

and from the maximality condition for H it follows that

$$\gamma(\cdot) \subset \{\lambda \in T^*M : h_1(\lambda) = h_2(\lambda) = 0\}. \tag{4.5}$$

If we denote $D_2 = \mathrm{span}(f_1, f_2)$, then (4.5) is equivalent to $\gamma(\cdot) \subset (D_2)^\perp$. Combining (4.4) and (4.5), we obtain

$$d_{\gamma(t)} h_i\big(\dot{\gamma}(t)\big) = 0, i = 1, 2. \tag{4.6}$$

[1] the meaning of the word “subsystem’ is that at any point q the set of its admissible velocities is a subset of the set of the admissible velocities of the original system at q.

Then from (4.4) and (4.6) it follows

$$\begin{aligned}&\{h_0,h_1\}\big(\gamma(t)\big)+\bar u_2(t)\{h_2,h_1\}\big(\gamma(t)\big)=0,\\&\{h_0,h_2\}\big(\gamma(t)\big)+\bar u_1(t)\{h_1,h_2\}\big(\gamma(t)\big)=0\end{aligned}\tag{4.7}$$

(here $\{h_i,h_j\}$ are Poisson brackets of the Hamiltonians h_i and h_j: $\{h_i,h_j\}=dh_j(\vec h_i)$). Now suppose that

$$\dim D_2^2=3\tag{4.8}$$

Then relations (4.7) implies that the extremals of (4.1), lying in $(D_2)^\perp\backslash(D_2^2)^\perp$, are exactly the integral curves of the vector field

$$\vec X=\vec h_0+\frac{\{h_0,h_2\}}{\{h_2,h_1\}}\vec h_1+\frac{\{h_1,h_0\}}{\{h_2,h_1\}}\vec h_2\tag{4.9}$$

(which is the Hamiltonian vector field, corresponding to the Hamiltonian $X=h_0+\dfrac{\{h_0,h_2\}}{\{h_2,h_1\}}h_1+\dfrac{\{h_1,h_0\}}{\{h_2,h_1\}}h_2$). Denote by V the affine subbundle of TM, defined by system (4.1) and $V(q)$ be the set of all admissible velocities of the system (4.1) at the point q,

$$V(q)=\{f_0(q)+u_1f_1(q)+u_2f_2(q):u_1,u_2\in\mathbb R\}.$$

Let $\pi:T^*M\mapsto M$ be the canonical projection. The set

$$\mathrm{Ext}(q)=\{\pi_*\vec X(\lambda):\lambda\in T_q^*M\cap(D_2)^\perp\backslash(D_2^2)^\perp\},\quad q\in M\tag{4.10}$$

is the subset of $V(q)$, consisting of the velocities of all extremal trajectories starting at q and having a lift in $(D_2)^\perp\backslash(D_2^2)^\perp$.

Among all extremals on $(D_2)^\perp\backslash(D_2^2)^\perp$, one can distinguish so-called abnormal extremals, i.e., the extremals lying on the zero level set of the Hamiltonian X. Denote $D_3=\mathrm{span}(f_0,f_1,f_2)$ and suppose that

$$\dim\,(D_2^2+D_3)=4\tag{4.11}$$

The set

$$\mathrm{Abn}(q)=\{\pi_*\vec X(\lambda):\lambda\in T_q^*M\cap(D_3)^\perp\backslash(D_2^2)^\perp\},\quad q\in M\tag{4.12}$$

is the subset of $\mathrm{Ext}(q)$, consisting of the velocities of all abnormal extremal trajectories starting at q and having a lift in $(D_2)^\perp\backslash(D_2^2)^\perp$. One can show that for generic affine systems of the type (4.1) $\mathrm{Ext}(q)=V(q)$ in the case $n\ge 5$ and $\mathrm{Abn}(q)=V(q)$ in the case $n\ge 6$. But in the case $n=4$ and $n=5$ either $\mathrm{Ext}(q)$ or $\mathrm{Abn}(q)$ (or both of them) define the proper subsystem of the original system (4.1). Moreover, it turns out that these subsystems are affine with scalar input, so one can apply the theory of the previous section. Now let us consider the cases $n=4$ and $n=5$ separately.

4.2 The case $n=4$. Let

$$[V,D_2](q)=\{[X,Y](q):\ X\in V,\ Y\in D_2,\ \text{are vector fields}\}.$$

It is not difficult to show that $[V,D_2](q)$ is a linear space and

$$[V,D_2](q)=\mathrm{span}\big(f_1(q),f_2(q),[f_1,f_2](q),[f_0,f_1](q),[f_0,f_2](q)\big)\tag{4.13}$$

The crucial observation is formulated in the following

Proposition 3 *The set* $\mathrm{Ext}(q)$ *is an affine line, provided that (4.8) holds and*

$$\dim [V, D_2](q) = 4 \tag{4.14}$$

Proof. Take some vector field f_3 such that the tuple (f_0, f_1, f_2, f_3) constitutes the frame on M. Denote by c_{ji}^k the structural functions of this frame, i.e., the functions, satisfying

$$[f_i, f_j] = \sum_{k=0}^{3} c_{ji}^k f_k. \tag{4.15}$$

Using the following well-known property of the Poison brackets

$$\{h_i, h_j\}(p, q) = p \cdot [f_i, f_j](q), \quad q \in M, p \in T_q^* M \tag{4.16}$$

and (4.9), one can easily obtain that

$$\mathrm{Ext}(q) = \Pi(q) \cap V(q), \tag{4.17}$$

where

$$\begin{aligned} \Pi(q) = \{ \big(c_{12}^0(q)\nu + c_{12}^3(q)\mu\big) f_0(q) + \big(c_{20}^0(q)\nu + \\ c_{20}^3(q)\mu\big) f_1(q) + \big(c_{01}^0(q)\nu + c_{01}^3(q)\mu\big) f_2(q) : \mu, \nu \in \mathbb{R}\} \end{aligned} \tag{4.18}$$

From assumption (4.14) and identity (4.13) it follows that $\Pi(q)$ is a plane. Assumption (4.8) implies that the plane $\Pi(q)$ is not parallel to the plane $V(q)$. Note also both $\Pi(q)$ and $V(q)$ belong to $D_3(q)$. Hence by (4.17) the set $\mathrm{Ext}(q)$ is an affine line. □

Consider the control system such that $\mathrm{Ext}(q)$ is its set of the admissible velocities at q. By Proposition 3 it is an affine system with scalar input. We call this system *the reduction of the four-dimensional control-affine system* (4.1). The following proposition gives another characterization of the reduction of the system (4.1):

Proposition 4 *Assume that the four-dimensional control-affine system (4.1) satisfies the conditions (4.8) and (4.14). Then the subsystem*

$$\dot{q} = g_0 + u g_1, \tag{4.19}$$

of (4.1) is its reduction if and only if

$$[g_0, g_1] \in D_2. \tag{4.20}$$

Proof. By definition, the system (4.19) is the reduction of (4.1) if and only if

$$\mathrm{Ext}(q) = \{g_0(q) + t g_1(q) : t \in \mathbb{R}\}. \tag{4.21}$$

On the other hand, on can take from the beginning $f_0 = g_0$ and $f_1 = g_1$. Then comparing (4.21) with (4.17) and (4.18) we obtain that the system (4.19) is the reduction of (4.1) if and only if $c_{01}^0 = c_{01}^3 = 0$, which is equivalent to $[g_0, g_1] \in \mathrm{span}(g_1, f_2) = D_2$. □

Corollary 2 *Assume that a four-dimensional affine control system (4.19) satisfies*

$$\dim \operatorname{span}\big(g_1, [g_1, g_0], \big[g_1, [g_1, g_0]\big], \big[g_0, [g_1, g_0]\big]\big) = 4 \tag{4.22}$$

Then any four-dimensional control-affine system with two-dimensional input, having the system (4.19) as its reduction, is feedback equivalent to the system

$$\dot q = g_0 + u_1 g_1 + u_2 [g_0, g_1], \quad u_1, u_2 \in \mathbb{R}. \tag{4.23}$$

Proof. First by assumption (4.22) and relation (4.13) the system (4.23) satisfies conditions (4.8) and (4.14) (where f_0, f_1, and f_2 are replaced by g_0, g_1 and $[g_0, g_1]$). Hence, by Proposition 3 the system (4.23) admits the reduction and by Proposition 4 this reduction is the system (4.19). On the other hand, suppose that some system (4.1) has the reduction (4.19). Then from the previous proposition $[g_0, g_1] \in \operatorname{span}(g_1, f_2)$. According to (4.22), g_1 and $[g_0, g_1]$ are linearly independent. Hence the system (4.1) is feedback equivalent to (4.23). □

According to the previous proposition, a generic four-dimensional control-affine system with two-dimensional input can be uniquely, up to a feedback transformation, recovered from its reduction. Suppose that the reduction (4.19) of the system (4.1) satisfies (4.22) and

$$\dim \operatorname{span}\big(g_0, g_1, [g_1, g_0], \big[g_1, [g_1, g_0]\big]\big) = 4. \tag{4.24}$$

Then we can apply to the system (4.19) all constructions of section 2. In particular, one can construct the tuple of the primary invariants of (4.19) at a given point, which are also feedback invariants of the original system (4.1). Note that the set of germs of systems of the type (4.1) having the reductions, which satisfies conditions (4.22) and (4.24), is generic. Combining Theorem 1, Corollary 2, and normal form (3.32), we obtain the following classification of generic germs of systems of the type (4.1) in terms of the tuple of the primary invariants of their reductions:

Theorem 3 *Given two arbitrary germs $\mathcal{I}_1$ and $\mathcal{I}_2$ of functions of four variables at 0, four arbitrary germs $\beta_1, \beta_2, \beta_3, \beta_4$ of functions of three variables at 0, and four arbitrary germs $\psi_1, \psi_2, \psi_3, \psi_4$ of functions of two variables at 0 there exists a unique, up to state-feedback transformation of the type (1.3), four-dimensional control-affine system with two-dimensional input such that its reduction satisfies genericity assumptions (4.22) and (4.24) and $(\mathcal{I}_1, \mathcal{I}_2, \beta_1, \beta_2, \beta_3, \beta_4, \psi_1, \psi_2, \psi_3, \psi_4)$ is the tuple of the primary invariants of the reduction at the given point q_0. This control system is state-feedback equivalent to the following one:*

$$\begin{cases} \dot x_1 = 1 + (N + B_1\rho_2)u_1 + \Big(\dfrac{\partial N}{\partial x_1} + \beta_1 \dfrac{\partial \rho_2}{\partial x_1}\Big)u_2 \\ \dot x_2 = (\rho_1 + B_2\rho_2)u_1 + \Big(\dfrac{\partial \rho_1}{\partial x_1} + B_2 \dfrac{\partial \rho_2}{\partial x_1}\Big)u_2 \\ \dot x_i = B_i\rho_2 u_1 + \beta_i \dfrac{\partial \rho_2}{\partial x_1} u_2, \quad i = 3, 4 \end{cases}, \quad u_1, u_2 \in \mathbb{R}, \tag{4.25}$$

where N is the solution of (3.29), ρ_i, $i = 1, 2$ are the solutions of (3.30), and B_k, $1 \le k \le 4$, are as in (3.31). The Poincare series, the characteristic pair and the characteristic matrix of the classification problem are exactly the same as in the case of $(1, 4)$ control-affine systems.

Remark 7 It is easy to show that in the case $n = 4$ the set $\operatorname{Abn}(q)$ consists of one vector provided that (4.11) holds. Besides, if the system (4.19) is the reduction of the system (4.1) and it satisfies (4.22) and (4.24), then $\operatorname{Abn}(q)$ is exactly its canonical drift.□

Remark 8 Actually, there is another intrinsic way to assign to the system (4.1), satisfying (4.11), an affine subsystem with scalar input: As a drift one can take again $Ab(q)$. It remains to define canonically the direction of the affine line of the reduction. For this note first that the distribution D_2 satisfies (4.8) because of assumption (4.11). Therefore through any point of M the unique (unparameterized) abnormal extremal trajectory of the rank 2 distribution D_2 passes: the line subdistribution L of D_2, tangent to the abnormal extremal trajectories at any point is characterized by the relation $[L, D^2] \subseteq D^2$. The direction of the affine line of the reduction can be taken parallel to L. The direction of L is different in general from the direction of the affine line in the first reduction. But this new reduction is worse than the previous one, because the original system (4.1) is not uniquely recovered from it: if $(\bar{g}_0, \bar{g}_1)$ is the canonical pair of the new reduction (by construction and the previous remark $g_0 = \mathrm{Abn}$), then the field f_2 can be taken in the form $f_2 = \alpha\bar{g}_0 + [\bar{g}_1, \bar{g}_0]$, where the function α satisfies some second order ordinary differential equation along each integral curve of $\bar{g}_1$. Note that the direction $\bar{g}_1$ depends on the second jet of the original system (4.1), while the direction of the affine line in the first reduction depends only on the first jet. This could be the reason for the loss of some information about the original system during the reduction described in the present remark. □

4.3 The case $n = 5$**.** In this case by analogy with Proposition 3 we have

Proposition 5 *The set* $\mathrm{Abn}(q)$ *is an affine line, provided that (4.11) holds and*

$$\dim D_3^2 = 5. \tag{4.26}$$

Proof. Take some vector fields f_3 and f_4 such that the tuple $(f_0, f_1, f_2, f_3, f_4)$ constitutes the frame on M. By analogy with (4.15), let c_{ji}^k, $0 \le i, j, k \le 4$, be the structural functions of this frame. From (4.9) and (4.12), using (4.16) and the fact that $h_0 = 0$ on $(D_3)^\perp$, one can easily obtain that

$$\mathrm{Abn}(q) = \Pi_1(q) \cap V(q), \tag{4.27}$$

where

$$\begin{aligned}\Pi_1(q) = \{&\big(c_{12}^3(q)\nu + c_{12}^4(q)\mu\big)f_0(q) + \big(c_{20}^3(q)\nu + \\ &c_{20}^4(q)\mu\big)f_1(q) + \big(c_{01}^3(q)\nu + c_{01}^4(q)\mu\big)f_2(q) : \mu, \nu \in \mathbb{R}\}\end{aligned} \tag{4.28}$$

From assumption (4.26) and identity (4.13) it follows that $\Pi_1(q)$ is a plane. Assumption (4.11) implies that the plane $\Pi_1(q)$ is not parallel to the plane $V(q)$. Note also both $\Pi_1(q)$ and $V(q)$ belong to $D_3(q)$. Hence by (4.27) the set $\mathrm{Abn}(q)$ is an affine line. □

Consider the control system such that $\mathrm{Abn}(q)$ is its set of the admissible velocities at q. By Proposition 5 it is an affine system with scalar input. We call this system *the reduction of the five-dimensional control-affine system* (4.1). The following proposition gives another characterization of the reduction of the system (4.1):

Proposition 6 *Assume that the five-dimensional control-affine system (4.1) satisfies the conditions (4.26) and (4.11). Then the subsystem*

$$\dot{q} = g_0 + ug_1, \tag{4.29}$$

of (4.1) is its reduction if and only if

$$[g_0, g_1] \in D_3. \tag{4.30}$$

Proof. By definition, the system (4.19) is the reduction of (4.1) if and only if

$$\mathrm{Abn}(q) = \{g_0(q) + tg_1(q) : t \in \mathbb{R}\}. \tag{4.31}$$

On the other hand, on can take from the beginning $f_0 = g_0$ and $f_1 = g_1$. Then comparing (4.31) with (4.27) and (4.28) we obtain that the system (4.29) is the reduction of (4.1) if and only if $c_{01}^3 = c_{01}^4 = 0$, which is equivalent to $[g_0, g_1] \in \mathrm{span}(g_0, g_1, f_2) = D_3$. □

Corollary 3 *Assume that a five-dimensional affine control system (4.19) satisfies*

$$\dim \mathrm{span}\big(g_0, g_1, [g_1, g_0], \big[g_1, [g_1, g_0]\big], \big[g_0, [g_1, g_0]\big]\big) = 5 \tag{4.32}$$

Then a five-dimensional control-affine system with two-dimensional input, has the system (4.19) as its reduction if and only if it is feedback equivalent to the system

$$\dot{q} = g_0 + u_1 g_1 + u_2(\alpha g_0 + [g_0, g_1]), \quad u_1, u_2 \in \mathbb{R}, \tag{4.33}$$

where α is some function.

Proof. First by assumption (4.32) the system (4.33) satisfies conditions (4.26) and (4.11). Hence, by Proposition 5 the system (4.33) admits the reduction and by Proposition 6 this reduction is the system (4.19). On the other hand, suppose that some system (4.1) has the reduction (4.19). Then from the previous proposition $[g_0, g_1] \in \mathrm{span}(g_0, g_1, f_2)$. According to (4.22), g_0, g_1 and $[g_0, g_1]$ are linearly independent. Hence, $f_2 = \xi_0 g_0 + \xi_1 g_1 + \xi_2 [g_0, g_1]$, where $\xi_2 \neq 0$. Hence by a feedback transformation we can replace f_2 with $\alpha g_0 + [g_0, g_1]$. □

So, in contrast to the case $n = 4$, five-dimensional control-affine systems with two-dimensional input cannot be recovered from its reduction only. Suppose that the system (4.1) has the reduction (4.29) satisfying condition (4.32) and also

$$\dim \mathrm{span}\big(g_0, g_1, [g_1, g_0], \big[g_1, [g_1, g_0]\big], \big[g_1, \big[g_1, [g_1, g_0]\big]\big]\big) = 5, \tag{4.34}$$

$$\dim \mathrm{span}\big(g_1, \big[g_0, [g_0, g_1]\big], [g_1, g_0], \big[g_1, [g_1, g_0]\big], \big[g_1, \big[g_1, [g_1, g_0]\big]\big]\big) = 5. \tag{4.35}$$

Then we can apply to the system (4.29) all constructions of section 2. In particular, let (G_0, G_1) be the canonical pair of the system (4.29). As before, let V be the affine subbundle of TM, defined by system (4.1). Then by the same arguments, as in the proof of Corollary 3, there exist a unique vector field $G_2 \in V$ such that

$$G_2 = RG_0 + [G_0, G_1]. \tag{4.36}$$

By construction, the function R is a feedback invariant of system (4.1). Moreover, the system (4.1) can be uniquely, up to a feedback transformation, recovered from its reduction and the function R.

Fix some point q_0 in M. Let Φ_5 be as in (3.17). Denote

$$\mathcal{R} = R \circ \Phi_5 \tag{4.37}$$

$\mathcal{R}$ is the germ of a function of five variables at 0. We call it *the recovering invariant of* (4.1) *at the point* q_0. By construction, it is invariant of the weight 0. Note that the set of germs of systems of the type (4.1) having the reductions, which satisfies conditions (4.32), (4.34), and (4.35), is generic. Using Theorem 2 in the case $n = 5$ and the definition of the recovering invariant, we obtain the following classification of generic real analytic germs of systems of the type (4.1) in terms of the tuple of the primary invariants of their reductions and their recovering invariant:

Theorem 4 *Given* 4 *germs* $\mathcal{I}_1$, $\mathcal{I}_2$, $\mathcal{I}_3$, *and* $\mathcal{R}$ *of real analytic functions of* 5 *variables at* 0 *such that* $\mathcal{I}_3(0) \neq 0$, 10 *germs* $\{\beta_k\}_{k=1}^5$ *and* $\{\phi_{k22}\}_{k=1}^5$ *of real analytic functions of* 4 *variables at* 0, 10 *germs* $\{\psi_k\}_{k=1}^5$ *and* $\{\phi_{k23}\}_{k=1}^5$ *of real analytic functions of* 3 *variables at* 0, *and* 5 *germs* $\{\phi_{k24}\}_{k=1}^5$ *of real analytic functions of* 2 *variables at* 0 *there exists a unique, up to state-feedback real analytic transformation of the type (1.3), five-dimensional real analytic control-affine system with two input such that first its reduction satisfies genericity assumptions (4.32),(4.34), (4.35), secondly* $\Big(\{\mathcal{I}_j\}_{j=1}^3, \{\beta_k\}_{k=1}^5, \{\psi_k\}_{k=1}^5, \{\phi_{k2l} : 1 \leq k \leq 5, 2 \leq l \leq 4\}\Big)$ *is its tuple of the primary invariants, and finally* $\mathcal{R}$ *is its recovering invariant at the given point* q_0. *In other words, the tuples of the primary invariants give the regular* $(3,2)$*-parameterization of the considered classification problem in the real analytic category with the following* 4×4 *parameterization matrix* P*:*

$$P = \begin{pmatrix} 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 10 \\ 0 & 0 & 0 & 10 \\ 1 & 0 & 0 & 3 \end{pmatrix}. \tag{4.38}$$

The Poincare series $M(t)$ *of the considered classification problem satisfies*

$$M(t) = \frac{1}{(1-t)^6} + t^3\left(\frac{5}{(1-t)^3} + \frac{10}{(1-t)^4} + \frac{10}{(1-t)^5} + \frac{3}{(1-t)^6}\right). \tag{4.39}$$

By the same arguments, as in the case of $(1,n)$-affine control system with $n \geq 5$, treated in section 3, one can show that (3,2) is the characteristic pair of the considered classification problem (just use formulas (2.24), (2.25), and Corollary 1). The characteristic matrix $\mathcal{C}$ of the problem is equal to $\mathrm{Norm}(P)$, which can be found easily by the series of elementary transformations. Namely,

$$\mathcal{C} = \mathrm{Norm}(P) = \begin{pmatrix} 1 & 3 & 6 & 5 \\ 0 & 0 & 0 & 16 \\ 0 & 0 & 0 & 13 \\ 0 & 0 & 0 & 4 \end{pmatrix}. \tag{4.40}$$

Conclusion 2 *The characteristic parameterization of* $(2,5)$ *control-affine systems, up to state-feedback transformations, consists of* 4 *functional invariants of* 5 *variables and the weight* 3, 13 *functional invariants of* 4 *variables and the weight* 3, 16 *functional invariants of* 3 *variables and the weight* 3, 1 *functional invariants of* 2 *variables and the weight* 0, 3 *functional invariants of* 2 *variables and the weight* 1, 6 *functional invariants of* 2 *variables and the weight* 2, *and* 5 *functional invariants of* 2 *variables and the weight* 3.

In order to obtain a characteristic parameterization from the parameterization by the tuple of the primary invariants and the recovering invariant one can implement some series of rearrangement of the primary invariants according to the series of elementary transformations from the matrix P to $\mathrm{Norm}(P)$, as was described in the previous section (see, for example, formula (2.19) and the paragraph after it).

References

[1] A.A. Agrachev, *Feedback-invariant optimal control theory - II. Jacobi Curves for Singular Extremals*, J. Dynamical and Control Systems, **4**(1998), No. 4 , 583-604.

[2] V.I. Arnold, *Mathematical problems in classical physics*, Trends and Perspectives in Applied Mathematics (Appl. Math. Series, V. 100), Editors: F.John,J.E. Marsden, L.Sirovich, New-York: Springer, 1994, 1-20; also in "Selected works of V.I. Arnold" (dedicated to his 60th anniversary, in Russian), 554-575.

[3] B. Jakubczyk, W. Respondek, *Feedback classification of analytic control systems in the plane*, in Analysis of Controlled Dynamical Systems, B. Bonnard et al., eds., Birkhäuser, Boston, 1991, pp. 262-273.

[4] W. Kang, *Extended controller form and invariants of nonlinear control systems with single input*, J. Math. System Estimation Control, 6(1996), pp. 27-51.

[5] W. Kang, A.J. Krener, Extended quadratic controller normal form and dynamic state feedback linearization of nonlinear systems, SIAM J. Control Optim., 30(1992), pp. 1319-1337.

[6] W. Respondek, *Feedback classification of nonlinear control systems in* $\mathbb{R}^2$ *and* $\mathbb{R}^3$, in Geometry of Feedback and Optimal Control, B. Jakubczyk and W. Respondek, eds., Marcel Dekker, New York, 1998, pp. 347-382.

[7] I. A. Tall, W. Respondek , *Feedback Classification of Nonlinear Single-Input Control Systems with Controllable Linearization: Normal Forms, Canonical Forms, and Invariants*, SIAM J. Control Optim., 41(2003), pp. 1498-1531.

[8] M. Zhitomirskii, *Typical Singularities of Differential 1-forms and Pfaffian equations*, Trans. Math. Monographs 113, Amer. Math. Soc., Providence, 1992, 176 p.

[9] M. Zhitomirskii, W. Respondek, *Simple germs of corank one affine distributions*, in Singularities Symposium-Lojasiewicz 70, Banch Center Pub. 44, B. Jakubczyk, W. Pawlucki, and J. Stasica, eds., Polish Acad. Sci., Warsaw, 1998, pp. 269-276.