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NONREGULAR ABNORMAL EXTREMALS OF 2-DISTRIBUTION: EXISTENCE, SECOND VARIATION, AND RIGIDITY

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ABSTRACT. We study existence and rigidity (W^1_∞-isolatedness) of nonregular abnormal extremals of completely nonholonomic 2-distribution (nonregularity means that such extremals do not satisfy the strong generalized Legendre–Clebsch condition). Introducing the notion of diagonal form of the second variation, we generalize some results of A. Agrachev and A. Sarychev about rigidity of regular abnormal extremals to the nonregular case. In order to reduce the second variation to the diagonal form, we construct a special curve of Lagrangian subspaces, a Jacobi curve. We show that certain geometric properties of this curve (like simplicity) imply the rigidity of the corresponding abnormal extremal.

1. INTRODUCTION

Let M be a smooth connected n-dimensional manifold. We say that D is a k-dimensional distribution on M, if for each point $q \in M$ a subspace $D(q)$ of the tangent space T_qM is chosen and $D(q)$ depends smoothly on q. In other words, k-dimensional distribution is a smooth k-dimensional subbundle of the tangent bundle TM. D can be defined locally by k smooth vector fields $f_1, \dots, f_k$ such that $D(q) = \mathrm{span}(f_1(q), \dots, f_k(q))$. A tuple of the fields $(f_1, \dots, f_k)$ is called a local basis of D. Denote by D^l the lth power of the distribution D, i.e., $D^l = [D, D^{l-1}]$. We will consider so-called completely nonholonomic (or bracket-generating) distributions. Distribution D is called completely nonholonomic, if for any $q \in M$ there exists an integer $l(q)$ such that $D^{l(q)}(q) = T_qM$.

A Lipschitzian curve $\gamma(\tau)$ is called admissible w.r.t. D, if the curve γ is tangent to D almost everywhere, i.e., $\dot\gamma(\tau) \in D(\gamma(\tau))$ for almost all τ. The set of all admissible curves $\gamma : [0, T] \to M$ can be endowed with the

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W^1_∞-topology. In a local basis $f_1, \dots, f_k$ to each admissible curve γ one can assign L_∞-functions (controls) $u_1, \dots, u_k$ such that $\dot\gamma = u_1 f_1 + \dots + u_k f_k$. Given a point q_0 and a time T denote by $\Omega_{q_0}(T)$ the set of all admissible curves $\gamma : [0,T] \to M$, starting at q_0 and by $\mathcal{F}_{q_0,T} : \Omega_{q_0}(T) \to M$ the endpoint mapping that takes each $\gamma \in \Omega_{q_0}(T)$ to the endpoint $\gamma(T)$. Note that the set $\Omega_{q_0}(T)$ has the structure of $(L_\infty[0,T])^k$-manifold (after choosing a local basis the corresponding controls define charts). An admissible curve $\gamma : [0,T] \to M$ is called an *abnormal extremal* of the distribution D, if it is a critical point of the mapping $\mathcal{F}_{\gamma(0),T}$, i.e., $\operatorname{Im} \mathcal{F}'_{\gamma(0),T}(\gamma) \neq T_{\gamma(T)}M$. The term abnormal extremal actually came from the Pontryagin maximum principle: defining on the set $\Omega^{q_1}_{q_0}(T)$ of all admissible curves $\gamma : [0,T] \to M$ joining q_0 to q_1 some functional of integral type (for example, length functional w.r.t. to some Riemannian metric on M), we get the optimal control problem. Abnormal extremals joining q_0 with q_1 are exactly the extremals of this problem with vanishing Lagrange multiplier near the functional. Abnormal extremal γ together with corresponding Lagrange multipliers gives an *abnormal lift* of γ to the cotangent bundle T^*M (or an *abnormal biextremal*).

The set $\Omega^{q_1}_{q_0}(T)$ is not empty by the well-known Rashevsky–Chow theorem, but this set may cease to be a Banach manifold in a neighborhood of an abnormal extremals joining q_0 to q_1. Moreover, an abnormal extremal γ can be in some sense an isolated point of $\Omega^{\gamma(T)}_{\gamma(0)}(T)$. More precisely, the intersection of some W^1_∞-neighborhood of γ in the space of all Lipschitzian curves with $\Omega^{q_1}_{q_0}(T)$ may contain only smooth reparametrizations of γ. Such γ is called *rigid curve* (note that only abnormal extremals can be rigid curves).

For 2-distributions a special characteristic vector field Ab on the annihilator $(D^2)^\perp \subset T^*M$ of D^2 is defined, up to multiplication by a function without zeros. The set of stationary points of Ab coincides with the annihilator $(D^3)^\perp \subset T^*M$ of D^3. The characteristic field Ab has the following property: all abnormal biextremals that are transversal to $(D^3)^\perp \subset T^*M$ geometrically coincide with integral curves of Ab. One can distinguish regular and nonregular abnormal biextremals: an abnormal biextremal is called regular, if it does not contain stationary points of Ab, and nonregular otherwise. The regular case was studied in the works [1], [9], [14]. In particular, it was proved that abnormal extremal corresponding to the regular abnormal biextremal is locally rigid, i.e., its restrictions to the sufficiently small time intervals are rigid.

Our paper is devoted to the nonregular abnormal biextremals: their existence and rigidity of the corresponding abnormal extremals. First results of this type were obtained for the case $n = 3$ in [15], using local normal forms of 2-distributions. In the present paper we treat the general n-dimensional case by another approach. The characteristic field Ab has 3 types of stationary points: elliptic, hyperbolic and parabolic, according to the eigenvalues of its linearization at these points. First, we prove the existence of the abnormal biextremal, passing through a hyperbolic point, and nonexistence of the abnormal biextremal, passing through an elliptic point. Second, we prove that if one endpoint of the abnormal biextremal is hyperbolic and all other points are regular (i.e., nonstationary points of the field Ab), then the corresponding abnormal extremal is locally rigid. We associate with such an abnormal biextremal a special curve of Lagrangian subspaces, Jacobi curve, in appropriate symplectic space. We give a sufficient condition for rigidity of the corresponding extremal in terms of this Jacobi curve.

Now few words about the method of proving the rigidity. The rigidity of the abnormal extremal γ is equivalent to the isolatedness of γ on the critical level set of appropriate endpoint mapping. Using the second differential at γ of this mapping, one can define a special quadratic form for any abnormal lift Γ of γ, called the second variation of Γ. This quadratic form can be written as follows:

$$\int_0^T \Big(P(\tau) w^2(\tau) + \Big(\int_0^\tau K(\tau, s) w(s)\, ds \Big) w(\tau) \Big) d\tau. \tag{1.1}$$

The scalar functions $P(\tau)$ and $K(\tau, s)$ depend on the basis of D in a neighborhood of γ. The function $P(\tau)$ has the following property: $P(\tau) = 0$ iff the point $\Gamma(\tau)$ is a stationary point of the field Ab. If P changes signs, it is similar to the absence of the Legendre condition in the Calculus of Variations. In this case, as a rule, the corresponding extremal is not locally rigid. If P does not change sign (for example, if abnormal biextremal is regular or contains stationary points of the field Ab only as endpoints), we would like to find a special basis of the distribution D such that

$$K(\tau, s) \equiv 0, \quad 0 \le \tau, s \le T. \tag{1.2}$$

Such a basis is called diagonalizing and we say that the second variation has in this basis diagonal form. If the diagonalizing basis exists, then the second variation becomes positive definite in some specially constructed Hilbert space. We connect the existence of the diagonalizing basis with some geometric properties (like the simplicity) of the Jacobi curve, mentioned above (Theorem 4.1). Finally, using the general theorem about the isolatedness of points on a critical levels of maps (see [1]) and making appropriate estimates of the second differential and the remainder terms in the Taylor expansion of the endpont mapping, we prove that the existence of the diagonalizing basis implies the rigidity (see Appendix B).

As far as we know, the idea to find a diagonalizing basis is new for such kind of problems (in the classical calculus of variations similar approach was considered by Clebsch). This is the main idea of our paper. Note also that the Jacobi curve that we obtain only in order to diagonalize the second variation, can be obtained, at least for the regular case, from the general theory of $\mathcal{L}$-derivarives, developed recently in [6] and [7] for studying feedback invariants of control systems.

Note further that the term "Jacobi curve for the regular abnormal extremal" in [1] (and earlier in [2]) used for similar but not the same object. In [1], [2] in order to compute the index of the second variation of the regular abnormal biextremal $\Gamma : [0,T] \to T^*M$ the authors introduce a special time-dependent linear Hamiltonian system in appropriate symplectic space (for example, in $T_{\Gamma(0)}T^*M$). By analogy with the classical calculus of variations this system is called Jacobi equation (note that for the non-regular case the Jacobi equation has singularities at the appropriate time moments). The Jacobi curve $J_\Gamma : [0,T] \to T_{\Gamma(0)}T^*M$ of Γ in [1], [2] is some discontinuous solution of the Jacobi equation with prescribed initial condition and jumps (in considered case actually with only one jump). Given any $t \in [0,T]$ denote by $\Gamma_t = \Gamma|_{[0,t]}$ the restriction of Γ to the interval $[0,t]$ and let $J_{\Gamma_t} : [0,t] \to T_{\Gamma(0)}T^*M$ be the Jacobi curve of Γ_t in the sense of [1], [2]. The curves J_{Γ_t} for different t satisfy the same Jacobi equation but have different initial conditions. It turns out that our Jacobi curve can be naturally identified (by appropriate factorization) with the curve $t \to J_{\Gamma_t}(t)$, i.e., with the curve consisting of endpoints of the Jacobi curves (in the sense of [1], [2]) of the restrictions Γ_t (see also [7]). In contrast to our Jacobi curve the Jacobi curve of [1], [2] depends on the local basis of the distribution.

2. Description of abnormal biextremals of 2-distribution

2.1. Characteristic field of abnormal biextremals. Writing the conditions of Pontryagin maximum principle in the abnormal case, one can get the description of abnormal extremals that gives a simple way to find these curves. We want to use a coordinate-free version of maximum principle (see [14]). For this reason we begin with some notations.

Let T^*M be the cotangent bundle of the manifold M and let $\pi : T^*M \to M$ be the canonical projection. We say that an absolutely continuous curve $\Gamma \subset T^*M$ is a lift of $\gamma \subset M$ if $\pi\Gamma = \gamma$. Let $\Gamma(t) = (\gamma(t), \lambda(t))$. The curve Γ is called nonzero lift of γ if $\lambda(t) \neq 0$ for any t.

The annihilator $(D^l)^\perp$ of the l-th power D^l of the distribution D is a subset of T^*M defined by

$$(D^l)^\perp = \{(q,\lambda) \in T^*M : \lambda \neq 0,\ \lambda \cdot v = 0 \ \ \forall v \in D^l(q)\}.$$

Let σ be the canonical symplectic form on T^*M. Denote by σ_1 the restriction of σ to $D^\perp$, $\sigma_1 = \sigma|_{D^\perp}$. Given a smooth vector field X, denote by H_X the corresponding Hamiltonian function $H_X(q,\lambda) = \lambda \cdot X(q)$ and by $\vec{H}_X$

the Hamiltonian lift of the field X, i.e., a vector field on T^*M, satisfying $\vec{H}_X \rfloor \sigma = -dH_X$. Now we are ready to formulate the coordinate-free version of Pontryagin maximum principle in the abnormal case (see [14], [9]):

Proposition 2.1. *Absolutely continuous curve $\gamma(t)$ is abnormal extremal iff there exists a nonzero lift $\Gamma(t) \subset T^*M$ of γ, satisfying the following 2 conditions:*
(1) $\Gamma(t)$ lies in $D^\perp$,
(2) $\dot{\Gamma}(t)$ belongs to $\operatorname{Ker}\sigma_1$ a.e.

Remark 2.1. The first condition on Γ in the previous proposition is actually the maximality condition for Hamiltonian, the second condition is the corresponding Hamilton system of equations.

The curve Γ of the previous proposition is called an *abnormal biextremal* or an *abnormal lift* of γ.

For any point $Q \in D^\perp$ one can define the lift $\vec{H}_D = \vec{H}_D(Q) \subset T_Q T^*M$ of the distribution D: $\vec{H}_D = \operatorname{span}\{\vec{H}_{f_1}, \dots \vec{H}_{f_k}\}$, where $(f_1, \dots, f_k)$ is a basis of D in some neighborhood of πQ in M. This definition does not depend on the choice of the basis $(f_1, \dots, f_k)$, since $\vec{H}_{\alpha_1 f_1 + \dots + \alpha_k f_k}(Q) = \alpha_1 \vec{H}_{f_1}(Q) + \dots + \alpha_k \vec{H}_{f_k}(Q)$ for any $Q \in D^\perp$, where $\alpha_1, \dots, \alpha_k$ are arbitrary smooth functions on M. We will need further also the following formula

$$dH_X \cdot \vec{H}_Y = H_{[Y,X]}. \tag{2.3}$$

Lemma 2.1. *For any $Q \in (D)^\perp$*

$$\operatorname{Ker}\sigma_1(Q) = \vec{H}_D(Q) \cap T_Q D^\perp. \tag{2.4}$$

Proof. $v \in \operatorname{Ker}\sigma_1(Q) \Leftrightarrow (v \rfloor \sigma)(Q)|_{D^\perp} = 0 \Leftrightarrow$ there exist numbers $\alpha_1, \dots, \alpha_k$ such that $v \rfloor \sigma(Q) = \alpha_1 dH_{f_1}(Q) + \dots + \alpha_k dH_{f_k}(Q) = -(\alpha_1 \vec{H}_{f_1}(Q) + \dots + \alpha_k \vec{H}_{f_k}(Q)) \rfloor \sigma(Q) \Leftrightarrow v = -(\alpha_1 \vec{H}_{f_1}(Q) + \dots + \alpha_k \vec{H}_{f_k}(Q)) \in \vec{H}_D(Q)$, since σ is not a degenerated form. $\square$

From now we consider only 2-distributions (i.e., $k = 2$). Denote by (f, g) a local basis of the 2-distribution D.

Proposition 2.2. *For 2-distribution D the form $\sigma_1(Q)$ (where $Q \in D^\perp$) is degenerated (i.e., has nontrivial kernel) iff $Q \in (D^2)^\perp$. Moreover, for any $Q \in (D^2)^\perp$*

$$\operatorname{Ker}\sigma_1 = \vec{H}_D. \tag{2.5}$$

Proof. From (2.4) it follows that

$$\begin{aligned}\operatorname{Ker}\sigma_1(Q) &= \vec{H}_D(Q) \cap T_Q D^\perp = \\ &= \operatorname{span}\{\vec{H}_f(Q), \vec{H}_g(Q)\} \cap \{v : dH_f(v) = dH_g(v) = 0\}.\end{aligned}$$

Thus, if nonzero vector v belongs to $\operatorname{Ker}\sigma_1(Q)$, then there exist numbers α and β ($\alpha^2+\beta^2 \neq 0$) such that $v = \alpha\vec{H}_f(Q) + \beta\vec{H}_g(Q)$ and $dH_f(Q)\cdot(\alpha\vec{H}_f(Q) + \beta\vec{H}_g(Q)) = dH_g(Q)\cdot(\alpha\vec{H}_f(Q) + \beta\vec{H}_g(Q)) = 0$. Note that $dH_f \cdot \vec{H}_f = dH_g \cdot \vec{H}_g = 0$. Using (2.3), we get that for the existence of the numbers α and β it is necessary that $H_{[f,g]}(Q) = -dH_f(Q)\cdot\vec{H}_g(Q) = 0$. Thus, $Q \in (D^2)^\perp$.

If $Q \in (D^2)^\perp$, then $dH_f(Q)\cdot\vec{H}_g(Q) = dH_g(Q)\cdot\vec{H}_f(Q) = 0 \Rightarrow \vec{H}_D(Q) \subset T_Q D^\perp$. This implies (2.5). $\square$

Propositions 2.1 and 2.2 yield the following corollary.

Corollary 2.1. *Any abnormal biextremal Γ of 2-distribution D lies in $(D^2)^\perp$. In addition, if $(D^2)^\perp$ is a smooth manifold, then*

$$\dot{\Gamma}(t) \subset \vec{H}_D(\Gamma(t)) \cap T_{\Gamma(t)}(D^2)^\perp \ a.e. \tag{2.6}$$

We distinguish two types of points in $(D^2)^\perp$: the point Q is called *regular*, if $Q \in (D^2)^\perp \setminus (D^3)^\perp$, and *nonregular*, if $Q \in (D^3)^\perp$. Note that Q is regular iff at least one of the numbers $dH_{[f,g]}\vec{H}_f(Q)$ and $dH_{[f,g]}\vec{H}_g(Q)$ is not equal to zero. First, this gives us that in this case the form $dH_{[f,g]}(Q)|_{T_Q D^\perp} \neq 0$, which implies that in a neighborhood of the regular point Q the set $(D^2)^\perp$ is automatically smooth submanifold of codimension 1 in $D^\perp$. Second, this yields that the point Q is regular iff $\vec{H}_D(Q)$ is *transversal to* $T_Q(D^2)^\perp$ *in* $T_Q D^\perp$. In the nonregular case the set $(D^2)^\perp$ is, in general, not a smooth manifold. From now we assume that $(D^2)^\perp$ is a smooth submanifold of codimension 1 in $D^\perp$ also at the nonregular points. In this case the point Q is nonregular iff $\vec{H}_D(Q) \subset T_Q(D^2)^\perp$ by the above. Note that condition (2.12) below implies the last assumption and that (2.12) holds for generic 2-distributions (see paragraph after Corollary 2.6 below). Hence, for the generic 2-distribution $(D^2)^\perp$ is a smooth submanifold of codimension 1 in $D^\perp$.

An abnormal biextremal Γ is called *regular*, if all its points $\Gamma(\tau)$ are regular points. An abnormal biextremal Γ is called *nonregular*, if it contains at least one nonregular point. The projections of regular and nonregular abnormal biextremals are called correspondingly *regular* and *nonregular* abnormal extremals (note that abnormal extremals can be regular and nonregular at the same time).

The intersection $\vec{H}_D(Q) \cap T_Q(D^2)^\perp$ defines a direction for any regular point Q. Therefore, one can define a characteristic field of directions on $(D^2)^\perp$, assuming that the nonregular points are its singular (stationary) points. Relation (2.6) shows that any regular abnormal biextremal of D is an integral curve of this field of directions.

Our aim is to describe the nonregular abnormal biextremal of D. First, let us define a characteristic vector field of D. By a direct computation, one can get the following relation for any regular point Q and any basis (f, g) of D in some neighborhood of πQ on M:

$$\vec{H}_D(Q) \cap T_Q(D^2)^\perp = \operatorname{span}\{H_{[f,[f,g]]}(Q)\vec{H}_g(Q) - H_{[g,[f,g]]}(Q)\vec{H}_f(Q)\}. \tag{2.7}$$

We say that an open set $V \subset (D^2)^\perp$ is *nice*, if a basis (f, g) of the distribution D can be chosen on the set $\pi V \subset M$. Denote by $Ab_{f,g}$ the following vector field on V:

$$Ab_{f,g} = H_{[f,[f,g]]}\vec{H}_g - H_{[g,[f,g]]}\vec{H}_f. \tag{2.8}$$

Note that the set of the stationary points of $Ab_{f,g}$ coincides with $(D^3)^\perp \cap V$. Suppose that $(\tilde{f}, \tilde{g})$ is another basis of the distribution D in πV, where $\tilde{f} = af + bg$ and $\tilde{g} = cf + dg$. By a direct computation we have

$$H_{[\tilde{f},[\tilde{f},\tilde{g}]]}\vec{H}_{\tilde{g}} - H_{[\tilde{g},[\tilde{f},\tilde{g}]]}\vec{H}_{\tilde{f}} = \Delta^2(H_{[f,[f,g]]}\vec{H}_g - H_{[g,[f,g]]}\vec{H}_f), \tag{2.9}$$

where $\Delta = ad - bc \neq 0$.

Definition 2.1. A vector field defined on the nice set V is called a characteristic vector field of the distribution D on V, if it coincides, up to multiplication by a function without zeros, with the field $Ab_{f,g}$ for some basis (f, g) of D on the projection πV.

Such "local" definition of the characteristic field is actually sufficient for our purposes. Anyway we want to show, how the characteristic field can be defined globally, i.e., on the whole $(D^2)^\perp$.

Proposition 2.3. *There exists a vector field Ab on $(D^2)^\perp$ such that the restriction of Ab to any nice set V is a characteristic field of D on V. The field Ab is unique up to multiplication by a function without zeros.*

Proof. The field Ab can be constructed with the help of the standard arguments of partition of unity. One can find a locally finite covering $\{V_\alpha\}$ of $(D^2)^\perp$ by nice sets. Let (f_α, g_α) be some basis of D on the set πV_α. By (2.9) for any α_1 and α_2 the vector fields $Ab_{f_{\alpha_1}, g_{\alpha_1}}$ and $Ab_{f_{\alpha_2}, g_{\alpha_2}}$ coincide on $V_{\alpha_1} \cap V_{\alpha_2}$ up to multiplication by a positive function ψ_{α_1,α_2}. Let $\{\phi_\alpha\}$ be the partition of unity, associated with the covering $\{V_\alpha\}$, then the field $Ab = \sum\limits_\alpha \phi_\alpha Ab_{f_\alpha, g_\alpha}$ is defined on the whole $(D^2)^\perp$ and for each α_0 $Ab = \Psi_{\alpha_0} Ab_{f_\alpha, g_\alpha}$, where $\Psi_{\alpha_0} = \sum_\alpha \psi_{\alpha_0,\alpha}\phi_\alpha > 0$. This implies that Ab is the vector field what we wanted to find. □

The vector field Ab from Proposition 2.3 is called a *characteristic vector field* of the distribution D. By the above, the set of stationary points of Ab coincides with $(D^3)^\perp$.

Denote $\sigma_2 = \sigma_1|_{(D^2)^\perp}$. For the regular points Q, using (2.5) and the transversality of $\vec{H}_D$ to $T_Q(D^2)^\perp$, we obtain that $\operatorname{Ker}\sigma_2(Q)$ is one dimensional and coincides with the direction of the characteristic vector field Ab at Q. Being the field of kernels of the closed form σ_2, the characteristic field is divergent free, therefore the sum of the eigenvalues of the linearization of Ab at any stationary point is equal to 0. Since codimension of $(D^3)^\perp$ in $(D^2)^\perp$ is equal to 2, there are not more than 2 nonzero eigenvalues. Thus, there are 3 cases:

(1) elliptic, when the nonzero eigenvalues of the linearization of Ab are purely imaginary and have opposite signs,

(2) hyperbolic, when the nonzero eigenvalues are real and have opposite signs,

(3) parabolic, when all eigenvalues are equal to 0.

The following lemma is needed for the sequel:

Lemma 2.2. *If the characteristic field coincides with $Ab_{f,g}$ in a neighborhood of the nonregular point Q, then nonzero eigenvalues of its linearization at Q are equal to the eigenvalues of the following 2×2-matrix:*

$$\begin{pmatrix} -H_{[f,[g,[f,g]]]} & -H_{[g,[g,[f,g]]]} \\ H_{[f,[f,[f,g]]]} & H_{[g,[f,[f,g]]]} \end{pmatrix} \tag{2.10}$$

Proof. Vector field of the form $\alpha_1 v_1 + \alpha_2 v_2$ (where α_1, α_2 are some functions and v_1, v_2 are independent vector fields) has in its stationary point (i.e., where $\alpha_1 = \alpha_2 = 0$) the following linearization A:

$$A = d\alpha_1 \otimes v_1 + d\alpha_2 \otimes v_2$$

(here $\xi \otimes v(w) = (\xi \cdot w)v$ for vectors v, w and functional ξ). Then $\operatorname{Im} A = \operatorname{span}(v_1, v_2)$, $\operatorname{Ker} A = (\operatorname{span}(d\alpha_1, d\alpha_2))^\perp$. By a direct computation, it can be shown that the nonzero eigenvalues are equal to the eigenvalues of 2×2-matrix $\begin{pmatrix} d\alpha_1 \cdot v_1 & d\alpha_1 \cdot v_2 \\ d\alpha_2 \cdot v_1 & d\alpha_2 \cdot v_2 \end{pmatrix}$ and the corresponding eigenvectors belong to $\operatorname{span}(v_1, v_2)$. Applying this to the field $Ab_{f,g}$ with

$$\alpha_1 = -H_{[g,[f,g]]}, \quad \alpha_2 = H_{[f,[f,g]]}, \quad v_1 = \vec{H}_f, \quad v_2 = \vec{H}_g \tag{2.11}$$

and using formula (2.3), one can prove the lemma. $\square$

Remark 2.2. From the Jacobi identity, it follows that

$$[f,[g,[f,g]]] + [g,[[f,g],f] = 0.$$

This is another proof of the fact that the sum of the eigenvalues of linearization of Ab at its stationary point is equal to 0.

Now we analyze separately the hyperbolic and the elliptic cases. We do not consider the parabolic case.

2.2. Hyperbolic case. The following proposition follows from the well-known theorems about invariant manifolds (see [10], p. 39; see also [8], p. 57 for the part (a)).

Proposition 2.4. (a) *For any hyperbolic point Q there are exactly two smooth invariant curves of the characteristic field, which contain this point and are transversal to $(D^3)^\perp$. One of these curves is a stable manifold of Q and the other is an unstable manifold.*

(b) *For each point of $(D^3)^\perp$ sufficiently close to Q we take the corresponding stable invariant curve, as described in part* (a). *Then the union of all such curves is a smooth manifold of codimension* 1 *in some neighborhood of Q in $(D^2)^\perp$. The same is true if we take unstable curves instead of stable.*

We call the invariant curves of the part (a) of the previous proposition separatrices of the hyperbolic point Q (this definition is slightly different from traditional terminology). Part (a) of the previous proposition and the fact that for the hyperbolic point $\vec{H}_D$ is transversal to $(D^3)^\perp$ in $(D^2)^\perp$ give the following description of abnormal biextremals, passing through a hyperbolic point.

Corollary 2.2. *If an abnormal biextremal $\Gamma : [0,T] \to (D^2)^\perp$ passes through a hyperbolic point Q, $Q = \Gamma(t_0)$, then there exists a neighborhood I_0 of t_0 in $[0,T]$ such that $\Gamma|_{I_0}$ belongs to the union of the two separatrices of the point Q. In other words, Γ_{I_0} geometrically coincides either with a part of one separatrix* (*smooth case*) *or with the union of parts of both separatrices of the point Q* (*nonsmooth case*).

Part (b) of Proposition 2.4 will be used in Sec. 4.

2.3. Elliptic case. An abnormal biextremal Γ is called trivial, if Γ is just a point, i.e., $\Gamma(t) \equiv Q$ for some $Q \in (D^2)^\perp$. The aim of this subsection is to prove the following proposition:

Proposition 2.5. *Suppose that for an elliptic point Q the vectors $\vec{H}_f(Q)$, $\vec{H}_g(Q)$, $\vec{H}_{[f,g]}(Q)$ are linearly independent. Then there exists a nontrivial abnormal biextremal passing through Q.*

Remark 2.3. The linear independence of the vectors $\vec{H}_f(Q)$, $\vec{H}_g(Q)$, $\vec{H}_{[f,g]}(Q)$ is equivalent to the following relation

$$dH_f \wedge dH_g \wedge dH_{[f,g]}(Q) \neq 0. \tag{2.12}$$

For the elliptic point $\vec{H}_D$ is transversal to $(D^3)^\perp$ in $(D^2)^\perp$ (therefore, by (2.6), nontrivial abnormal biextremal passing through Q cannot "stay" in $(D^3)^\perp$). This implies the following lemma.

Lemma 2.3. *Suppose that all points in a neighborhood U in $(D^3)^\perp$ of the elliptic point Q are not α- or ω-limits of any integral curve of the characteristic field Ab. Then there are no nontrivial abnormal biextremals passing through Q.*

Proof. Suppose that the converse is true. It means that there exists a nontrivial abnormal biextremal $\Gamma : [0,T] \to (D^2)^\perp$ passing through Q, $Q = \Gamma(t_0)$. Without loss of generality it can be assumed that all points of the neighborhood U are elliptic. Let us take a neighborhood V of Q in $(D^2)^\perp$ such that $V \cap (D^3)^\perp \subset U$. Then there exists a neigborhood (interval) I of t_0 in $[0,T]$ such that the restriction $\Gamma|_I$ of the curve Γ to I lies in V. Since Γ is nontrivial, one can choose the interval I such that $\Gamma|_I$ is nontrivial.

Suppose that $I_{\text{reg}} = \{t \in I : \Gamma(t) \text{ is regular}\}$. First we will show that the set I_{reg} is not empty. Indeed, if all points of $\Gamma|_I$ are nonregular, then $\dot\Gamma(t) \subset T_{\Gamma(t)}(D^3)^\perp$ for a.e. $t \in I$. On the other hand, by (2.6), it follows that $\dot\Gamma(t) \subset \vec{H}_D(\Gamma(t))$ for a.e. t. By the assumption $\Gamma(t)$ is an elliptic point (since $\Gamma(t) \subset U$). Hence the plane $\vec{H}_D(\Gamma(t))$ is transversal to $T_{\Gamma(t)}(D^3)^\perp$ in $T_{\Gamma(t)}(D^2)^\perp$. This yields that $\dot\Gamma(t) = 0$ for a.e. $t \in I$ that contradicts the assumption of nontriviality of the $\Gamma|_I$.

Thus, the set I_{reg} is a nonempty open subset of I. Let I_0 be a connected component of I_{reg}. Then the curve $\Gamma|_{I_0}$ is an integral curve of the field of directions of Ab and at least one of the endpoints Q_0 of I_0 is a nonregular (even elliptic) point contained in U. Thus, Q_0 is α- or ω-limit of the integral curve of Ab, which geometrically coincides with $\Gamma|_{I_0}$. This contradicts the assumptions of the lemma. $\square$

By the previous lemma, it follows that for proving Proposition 2.5 it is sufficient to show that Q and all points of $(D^3)^\perp$ sufficiently close to Q are not α- or ω-limits of any integral curve of the characteristic field Ab. Let us forget for a while about the special features of the construction of characteristic field Ab and consider the following problem:

Let v be a smooth vector field on the m-dimensional manifold N ($m > 2$), satisfying the following conditions:

(1) the set of the stationary points of v is a submanifold of N of codimension 2,

(2) the nonzero eigenvalues of the linearization of v at any stationary point are purely imaginary and have opposite signs.

Under what conditions stationary point Q is not α- or ω-limit of any integral curve of v?

First of all, diagonalizing simultaneously linear parts of v at all stationary points close to Q and "killing" the nonresonance terms up to the order 3, one can find the coordinates $(x_1, x_2, y_1, \dots, y_{n-2})$ in a neighborhood of Q such that the field v can be expressed by the following system of differential equations:

$$\begin{aligned} \dot{x}_1 &= \omega(y)x_2 + 2a(y)x_1R^2 - b(y)x_2R^2 + o(R^3), \\ \dot{x}_2 &= -\omega(y)x_1 + 2a(y)x_2R^2 + b(y)x_1R^2 + o(R^3), \\ \dot{y}_i &= \phi_i(y)R^2 + o(R^2), \end{aligned} \tag{2.13}$$

where $Q = 0$, $y = (y_1, \dots, y_{m-2})$, $R = \sqrt{x_1^2 + x_2^2}$, $i = 1, \dots, m-2$, $\omega(y)$, $a(y)$, and $b(y)$ are smooth functions, $\omega(y)$ is a nonzero eigenvalue of the linearization of v at the point with coordinates $(0, 0, y)$, $\omega(0) \neq 0$.

Lemma 2.4. *If at least one of the numbers $\phi_i(0)$ in (2.13) is not equal to 0, then the point 0 is not α- (or ω-) limit of any integral curve of v.*

Proof. Let for simplicity $\phi_1(0) \neq 0$. Suppose on the contrary that there exists an integral curve $\gamma(t)$ such that $\gamma(t) \underset{t\to+\infty}{\longrightarrow} 0$ (for the case $t \to -\infty$ the arguments are similar). From the first two equations of (2.13) it follows that

$$\dot{R} = a(y)R^3 + 0(R^3).$$

Then there exists a constant $C > 0$ such that

$$\left| \frac{d}{dt}\left(\frac{1}{R^2}\right) \right| = \left| a(y) + o(R) \right| < C \quad \forall T \geq 0.$$

Therefore, $R^2(t) > \dfrac{1}{C_1 + 2Ct}$, where $C_1 = \dfrac{1}{R^2(0)}$.

Let us consider the third equation of (2.13):

$$\dot{y}_1 = (\phi_1(y) + o(1))R^2. \tag{2.14}$$

The condition $\phi_1(0) \neq 0$ implies that for some neighborhood U of Q there exists a positive constant C_2 such that the right part of (2.14) is greater than C_2R^2. Since by the assumption $\gamma(t) \underset{t\to+\infty}{\longrightarrow} 0$, $\gamma(t) \in U$ for sufficiently large t, $\dot{y}_1 > C_2R^2 > \dfrac{C_2}{C_1 + 2Ct} \Rightarrow y_1(t) \underset{t\to+\infty}{\longrightarrow} \infty$. We have a contradiction. □

Now we will express the condition of nonvanishing of one of the numbers $\phi_i(0)$ in invariant terms. Let A be linearization of the field v in the point Q, A is actually linear operator, acting in T_QM. $\operatorname{Im} A$ denotes the image of the operator A, $\pm i\omega$ are nonzero eigenvalues of A.

Definition 2.2. A pair of vector fields (e_1, e_2) is called compatible with the field v in the point Q, if $e_1(Q) \in \operatorname{Im} A$, $e_1(Q) \neq 0$, and $Ae_1(Q) = \omega e_2(Q)$.

For any such pair we define a subspace of tangent space $T_Q M$ in the following way:

$$L = \operatorname{span}\left\{\operatorname{Im} A, ([e_1,[e_1,v]]+[e_2,[e_2,v]]-2\omega[e_1,e_2])(Q)\right\}.$$

Lemma 2.5. *The subspace L does not depend on the choice of the pair of the fields, compatible with the field v in the point Q, i.e., L is an invariant of v in Q.*

Proof. Let us introduce some coordinates $(x_1,\dots,x_m)$ in a neighborhood of Q. Let $v = v_i\frac{\partial}{\partial x_i}$, $e_k = e_{k,i}\frac{\partial}{\partial x_i}$ in these coordinates, where $k = 1,2$ (for the convenience, we omit here and below the sign of summation).

Taking into account that $v(Q) = 0$, we have:

$$\begin{aligned}
&[e_1,[e_1,v]]+[e_2,[e_2,v]]-2\omega[e_1,e_2] = \\
&=\left(\frac{\partial^2}{\partial x_i \partial x_k}v_j(e_{1,i}e_{1,k}+e_{2,i}e_{2,k})\frac{\partial}{\partial x_j}+\right.\\
&+\frac{\partial}{\partial x_i}v_j\left(\frac{\partial}{\partial x_k}e_{1,i}e_{1,k}+\frac{\partial}{\partial x_k}e_{2,i}e_{2,k}\right)\frac{\partial}{\partial x_j}-\\
&\left.-2\omega\left(\frac{\partial}{\partial x_i}e_{1,j}\frac{\partial}{\partial x_k}v_i e_{1,k}+\frac{\partial}{\partial x_i}e_{2,j}\frac{\partial}{\partial x_k}v_i e_{2,k}\right)\frac{\partial}{\partial x_j}\right)-\\
&-2\omega[e_1,e_2].
\end{aligned} \tag{2.15}$$

Note that $\frac{\partial}{\partial x_k}v_i(Q)e_{1,k}(Q) = Ae_1(Q) = \omega e_2(Q)$ and $\frac{\partial}{\partial x_k}v_i(Q)e_{2,k}(Q) = Ae_2(Q) = -\omega e_1(Q)$. Therefore, the third term of sum (2.15) at the point Q is canceled by the forth term. On the other hand, the second term of (2.15) at Q belongs to $\operatorname{Im} A$ and the first term at Q depends only on the values of the vectors $e_1(Q)$ and $e_2(Q)$. This proves that the subspace L does not depend, in fact, on the fields e_1 and e_2, but depends only on the vector $e_1(Q)$ (that defines correspondingly the vector $e_2(Q)$, $e_2(Q) = Ae_1(Q)$).

Let us show now that L does not depend also on $e_1(Q)$. It is sufficient to show that L does not change if we replace the pair (e_1,e_2) by the pair $(\tilde{e}_1,\tilde{e}_2) = (\lambda_1 e_1+\lambda_2 e_2, -\lambda_2 e_1+\lambda_1 e_2)$, where λ_1 and λ_2 are arbitrary constants, nonvanishing simultaneously (the pairs of fields of such type cover all possible values of $e_1(Q)$). By a direct computation $[\tilde{e}_1,[\tilde{e}_1,v]]+[\tilde{e}_2,[\tilde{e}_2,v]]-2\omega[\tilde{e}_1,\tilde{e}_2] = (\lambda_1^2+\lambda_2^2)([e_1,[e_1,v]]+[e_2,[e_2,v]]-2\omega[e_1,e_2])$. Thus, L does not depend on $e_1(Q)$. □

Corollary 2.3. *If the field v is represented in some coordinates by Eq. (2.13), then one of the numbers $\phi_i(0)$ is not equal to 0 if and only if $L \neq \operatorname{Im} A$.*

Proof. One can take $e_1 = \dfrac{\partial}{\partial x_1}$ and $e_2 = \dfrac{\partial}{\partial x_2}$. The pair (e_1, e_2) is compatible with v and

$$([e_1,[e_1,v]] + [e_2,[e_2,v]] - 2\omega[e_1,e_2])(Q) = (0,0,\phi_1(0),\dots,\phi_{m-2}(0)). \qquad \square$$

This corollary and Lemma 2.4 imply:

Corollary 2.4. *If $L \neq \operatorname{Im} A$, then the point Q is not α- (or ω-) limit of any integral curve of v.*

Now we suppose that the field v is represented in the form $v = \alpha_1 e_1 + \alpha_2 e_2$, where e_1 and e_2 are two independent vector fields, α_1 and α_2 are some functions (by assumptions on v, the set $\{\alpha_1 = 0\} \cap \{\alpha_2 = 0\}$ is a manifold of codimension 2, containing Q).

Lemma 2.6. *If the field v is represented in the form $v = \alpha_1 e_1 + \alpha_2 e_2$, then the invariant subspace L in Q coincides with the space $\tilde{L} = \operatorname{span}\{e_1(Q), e_2(Q), [e_1,e_2](Q)\}$.*

Proof. It is clear that the subspace $\tilde{L}$ does not change, if instead of the fields e_1 and e_2 one take other fields, generating the same 2-distribution, as the fields e_1 and e_2. Therefore, it is sufficient to prove the lemma in the case when $Ae_1(Q) = \omega e_2(Q)$. By Proposition 2.10, it follows that $A = d\alpha_1 \otimes e_1 + d\alpha_2 \otimes e_2 \Rightarrow d\alpha_1(e_1)e_1 + d\alpha_2(e_1)e_2 = \omega e_2 \Rightarrow d\alpha_2(e_1) = \omega$. Similarly $d\alpha_1(e_2) = -\omega$. Since $\alpha_1(Q) = \alpha_2(Q) = 0$, the vector $([e_1,[e_1,\alpha_1 e_1 + \alpha_2 e_2]] + [e_2,[e_2,\alpha_1 e_1 + \alpha_2 e_2]] - 2\omega[e_1,e_2])(Q) \in \tilde{L}$, and $L \subset \tilde{L}$. At the same time, opening the brackets in the last expression, one can easily obtain that the coefficient by $[e_1,e_2]$ is equal to $2(d\alpha_2(e_1) + d\alpha_1(e_2) - \omega) = -2\omega \neq 0$. Thus, $L = \tilde{L}$. $\square$

Proof of Proposition 2.5. The field Ab in a neighborhood of Q in $(D^2)^\perp$ coincides up to multiplication by function without zeros with $Ab_{f,g}$ for some basis (f,g) of D on πV. The field $Ab_{f,g}$ has the form $\alpha_1 v_1 + \alpha_2 v_2$ with α_1, α_2, v_1, and v_2 as in (2.11). Note that $[\vec{H}_f, \vec{H}_g] = \vec{H}_{[f,g]}$. The point Q and all points of $(D^3)^\perp$ that are sufficiently close to Q satisfy the conditions of Lemma 2.6. By Lemma 2.3, it follows that there are no nontrivial abnormal biextremals passing through Q. $\square$

Note that if $\dim D^2(\pi Q) = 3$, then the vectors $\vec{H}_f(Q)$, $\vec{H}_g(Q)$, and $\vec{H}_{[f,g]}(Q)$ are linearly independent automatically. Hence we have:

Corollary 2.5. *If Q is an elliptic point and $\dim D^2(\pi Q) = 3$, then there are no nontrivial abnormal biextremals, passing through Q.*

What happens in the case $\dim D^2(\pi Q) = 2$? It is not hard to see that the solvability of the equation $dH_f \wedge dH_g \wedge dH_{[f,g]} = 0$ is the condition of codimension $n+1$ on the 3-jet of the distribution D. Thus, by Thom transversality theorem, it follows that for the generic 2-distribution D all nonregular points on $(D^2)^\perp$ satisfy condition (2.12). Therefore, we have the following

Corollary 2.6. *For the generic 2-distribution there are no nontrivial abnormal biextremals passing through elliptic points.*

Remark 2.4. The result of this section about hyperbolic and elliptic points can also be proved, using Roussarie's normal forms for closed differential 2-forms obtained in [12] (see [16] for details).

Now let us discuss the case $n = 3$. In this case the image of $(D^2)^\perp$ under the projection π is the set of points $q \in M$ with $\dim D^2(q) = 2$. This set is called *Martinet surface*. In the local basis (f, g) of D the set S can be described by the equation $\det(f, g, [f, g]) = 0$. By a direct computation, it can be shown that linear independence of $\vec{H}_f(Q)$, $\vec{H}_g(Q)$, and $\vec{H}_{[f,g]}(Q)$ is equivalent to the following relation

$$d(\det(f, g, [f, g])(\pi Q) \neq 0. \tag{2.16}$$

Assuming that (2.16) holds for the nonregular Q (for the regular Q it holds automatically), we have that S is a smooth surface in M. Since $\pi_* \vec{H}_D(Q) = D(\pi Q)$, one can define the characteristic field of directions L on S: if Q is a regular point in $(D^2)^\perp$, then $D(\pi Q)$ is transversal to S in M and the intersection $D(\pi Q) \cap T_{\pi Q} M$ defines the direction of L at πQ; if Q is a nonregular point in $(D^2)^\perp$, then $D(\pi Q) = T_{\pi Q} S$ and the point πQ is by definition a singular (stationary) point of L. The characteristic vector field C of D on S can be defined in appropriate way, using projection of the characteristic field of D on $(D^2)^\perp$ to S. With the help of local normal form of 2-distribution (see [17], § 19) one can show that if Q is an elliptic point and condition (2.16) holds at πQ, then πQ is a stable or unstable point of the field C. Moreover, πQ is so-called weak degenerated focus of the field C and the trajectories that have πQ as their α- or ω-limit are spirals with infinite length w.r.t. any Riemannian metric on S. That was our argument in [15] to show that such spirals cannot be the abnormal extremals. Proposition 2.5 gives another explanation to this fact: the λ-component of the lift to $(D^2)^\perp$ of those spiral does not converge to a finite value (and actually tends to infinity) when one moves along the spiral towards the point πQ.

3. Diagonal form of the second variation

In this section we begin the investigation of the local rigidity of smooth abnormal extremals. Let $\gamma : [0, T] \to M$ be a smooth abnormal extremal such that $\dot{\gamma}(\tau) \neq 0$ for all $\tau \in [0,T]$, $\gamma(0) = q_0$ and $\gamma(T) = q_1$. Let $\Gamma : [0,T] \to (D^2)^\perp$ be a smooth abnormal lift of γ, $\Gamma(\tau) = (\gamma(\tau), \lambda(\tau))$. For simplicity, suppose also that γ has no self-intersections (i.e., $\gamma(\tau_1) \neq \gamma(\tau_2)$ for $\tau_1 \neq \tau_2$). A pair of vector fields (f,g) will be called a *local basis of the distribution D related to γ*, if (f,g) is a basis of D in some neighborhood of γ and γ is an integral curve of f. Given a local basis (f,g) related to γ, define a new endpoint mapping $F : L_\infty([0,T]) \to M$ in the following way: F maps a "control" $v(\cdot)$ into the point $q(T)$ of the trajectory $q(\cdot)$ of the system

$$\dot{q}(\tau) = f(q(\tau)) + v(\tau)g(q(\tau)), \quad q(0) = q_0. \tag{3.1}$$

The curve γ corresponds to the control $\hat{v}(\cdot) \equiv 0$ and abnormality of γ implies that $\hat{v}(\cdot)$ is a critical point of F.

Since γ has no self-intersections, in some neighborhood of γ one can choose a coordinate system $(x_1, \ldots, x_n)$ such that $f = \dfrac{\partial}{\partial x_1}$. Let $g = \sum_{i=1}^n \phi_i \dfrac{\partial}{\partial x_i}$ in these coordinates. Replacing g by the field $g - \phi_1 f$, one can make $\phi_1 \equiv 0$. Therefore, the local basis (f,g) of D related to γ can be chosen such that in some coordinates near γ

$$f = \frac{\partial}{\partial x_1}, \quad g = \sum_{i=2}^n \phi_i \frac{\partial}{\partial x_i} \tag{3.2}$$

for some smooth functions ϕ_i, $i = 2, \ldots, n$. The following proposition gives a description of rigidity in terms of the corresponding endpoint mapping:

Proposition 3.1. *The curve γ is rigid iff $\hat{v}$ is an isolated point on the critical level $F^{-1}(q_1)$ in $L_\infty([0,T])$ for the endpoint mapping F, corresponding to the local basis (f,g) of the form* (3.2).

Proof. The necessity follows directly from the definition of the rigidity. Let us prove the sufficiency. Suppose that $\hat{v}$ is an isolated point of $F^{-1}(q_1)$, then for some ε and any control $v(\cdot) \neq \hat{v}(\cdot)$ with $\|v\|_\infty < \varepsilon$ we have $F(v(\cdot)) \neq q_1$. Suppose that the basis (f,g) has the form (3.2) in the coordinates $(x_1, \ldots, x_n)$, defined in some neighborhood of γ, and $\gamma(t) = (t, 0, \ldots, 0)$. Take the curve $\tilde{\gamma}$ such that $\dot{\tilde{\gamma}} = u_1 f + u_2 g$, $\tilde{\gamma}(0) = q_0$, $\tilde{\gamma}(T) = q_1$, and $\|u_1(\cdot) - 1\|_\infty + \|u_2(\cdot)\|_\infty < \dfrac{\varepsilon}{\varepsilon+1}$. Then $\|u_2(\cdot)\|_\infty < \dfrac{\varepsilon}{\varepsilon+1}$ and $\|u_1(\cdot)\|_\infty > \dfrac{1}{\varepsilon+1}$. By the assumption, if $\tilde{\gamma}(t) = (x_1(t), \ldots, x_n(t))$, then $\dot{x}_1(t) = u_1(t)$. Define $\gamma_1(\tau) = \tilde{\gamma}(x_1^{-1}(\tau))$, then $\dot{\gamma}_1(\tau) = f(\gamma_1(\tau)) + \dfrac{u_2(x_1^{-1}(\tau))}{u_1(x_1^{-1}(\tau))} g(\gamma_1(\tau))$.

Note that $\gamma_1(T) = \tilde\gamma(T) = q_1$ and $\left\| \dfrac{u_2(\cdot)}{u_1(\cdot)} \right\|_\infty < \varepsilon$. Therefore, $\gamma_1 = \gamma$, i.e., the curve $\tilde\gamma$ is a reparametrization of γ. Thus, γ is rigid. □

From now, we consider only local bases (f, g) represented in some coordinates near γ in the form (3.2). Isolatedness of the point on the critical level of the mapping F is connected with properties of the second differential of F in this point. The quadratic form $\lambda(T) \cdot F''_{\hat v(\cdot)} v(\cdot)$ restricted to $\operatorname{Ker} F'_{\hat v(\cdot)}$ is called *the second variation of the abnormal biextremal* Γ *w.r.t. the local basis* (f, g). Note that, for rigid abnormal extremal of corank 1 (i.e., with codimension of $\operatorname{Im} F'$ in $T_{q_1}M$ equal to 1), the second variation of its lift is necessary either nonnegative or nonpositive (see [3], and also [1] for a generalization). In what follows we will prove that the curve γ is rigid, if the second variation w.r.t. some basis of one of its abnormal lifts Γ has a special form (Theorem 3.1).

Expressions for the first differential $F'_{\hat v(\cdot)}$ and the second variation of Γ obtained in [4] (see also [1]) with the help of chronological calculus, developed in [5] (see also Appendix A for a short survey of some ideas of chronological calculus). To write these expressions, let us introduce some notations. Let e^{tf} be the flow, generated by the vector field f. Any diffeomorphism P of M induces operator $\operatorname{Ad} P = (P^{-1})_*$ in the space of all vector fields on M. Denote $Y_\tau = \operatorname{Ad} e^{(\tau - T)f}$ and $Z_\tau = \operatorname{Ad} e^{\tau f}$. It is easy to prove that

$$\dot Y_\tau X = Y_\tau [f, X], \quad \dot Z_\tau X = Z_\tau [f, X] \tag{3.3}$$

for any vector field X, where "dot" means the derivative with respect to τ. Let $w(\cdot) = \int_0^{\cdot} v(\tau)\, d\tau$. Then

$$\begin{aligned} F'_{\hat v(\cdot)} v(\cdot) &= \int_0^T \operatorname{Ad} e^{(\tau - T)f} g(q_1) v(\tau)\, d\tau = \\ &= g(q_1) w(T) - \int_0^T Y_\tau [f, g](q_1) w(\tau)\, d\tau, \end{aligned} \tag{3.4}$$

$$\lambda(T)\cdot F''_{\hat v(\cdot)}v(\cdot)=\frac{1}{2}\lambda(T)\cdot\Bigg(\int\limits_0^T[\dot Y_\tau g,Y_\tau g](q_1)w^2(\tau)\,d\tau+$$
$$+\int\limits_0^T d\tau\,w(\tau)\int\limits_0^\tau[\dot Y_s g,\dot Y_\tau g](q_1)w(s)\,ds+$$
$$+w(T)\int\limits_0^T[g,\dot Y_\tau g](q_1)w(\tau)\,d\tau\Bigg),\tag{3.5}$$

(in the last expression $v(\cdot)$ belongs to $\operatorname{Ker}F'_{\hat v(\cdot)}$). Note that actually formulas (3.4) and (3.5) are figured out implicitly also in Appendix B.

In addition, it can be shown that the curve $\lambda(\tau)$ of covectors satisfies

$$\lambda(\tau)=(e^{-\tau f})^*\lambda(0)=(e^{(T-\tau)f})^*\lambda(T).\tag{3.6}$$

Now we want to single out the class of local bases, for which expression (3.5) looks more simple.

Definition 3.1. A local basis (f,g) of the distribution D related to γ is called quasi-normal if in some coordinates near γ we have

$$\begin{aligned}f&=\frac{\partial}{\partial x_1},\\ g&=\frac{\partial}{\partial x_2}+\sum_{i=3}^n\phi_i\frac{\partial}{\partial x_i}.\end{aligned}\tag{3.7}$$

Substituting the basis (f,g) of the form (3.7) in (3.4), we get that the x_1-component of $F'_{\hat v(\cdot)}v(\cdot)$ is equal to $w(T)$. Therefore, we have the following lemma.

Lemma 3.1. *If endpoint mapping F corresponds to a quasi-normal basis, then for any $v(\cdot)\in\operatorname{Ker}F'_{\hat v(\cdot)}$ we have $w(T)=0$.*

Consequently, for the endpoint mapping F, corresponding to the quasi-normal basis (f,g), the third term in (3.5) is equal to 0. Combining this fact with (3.6), we obtain that for all $v(\cdot)\in\operatorname{Ker}F'_{\hat v(\cdot)}$

$$2\lambda(T)\cdot F''_{\hat v(\cdot)}v(\cdot)=\int\limits_0^T\Bigg(P(\tau)w^2(\tau)+\Bigg(\int\limits_0^\tau K(\tau,s)w(s)\,ds\Bigg)w(\tau)\Bigg)\,d\tau,\tag{3.8}$$

where

$$P(\tau)=\lambda(\tau)\cdot[[f,g],g](\gamma(\tau))=H_{[[f,g],g]}(\Gamma(\tau)),\tag{3.9}$$

$$K(\tau, s) = \lambda(T) \cdot \Big[Y_s\,[f,g], Y_\tau\,[f,g]\Big](q_1) = \\ = \lambda(0) \cdot \Big[Z_s\,[f,g], Z_\tau\,[f,g]\Big](q_0). \tag{3.10}$$

Differentiating the identity $\lambda(\tau) \cdot [f,g](\gamma(\tau)) = 0$ with respect to τ and using the fact that γ is an integral curve of f, we have that

$$H_{[f,[f,g]]}(\Gamma(\tau)) = \lambda(\tau) \cdot [f,[f,g]](\gamma(\tau)) = 0. \tag{3.11}$$

Thus, $\Gamma(\tau)$ is a regular point iff $P(\tau) \neq 0$. It follows that for regular abnormal biextremals $P(\tau) \neq 0$ for all τ. Without loss of generality it can be assumed in this case that $P(\tau) > c > 0$ (this condition is similar to the strong Legendre condition). Then, using Cauchy–Schwartz inequality, one can prove that for sufficiently small T there exists a positive constant β, such that

$$\lambda(T) \cdot F''_{\tilde{v}(\cdot)}v(\cdot) > \beta \int_0 Tw^2(\tau)\,d\tau. \tag{3.12}$$

Now we suppose that for some τ_0 the point $\Gamma(\tau_0)$ is hyperbolic, then $P(\tau_0) = 0$. By a direct computation, $P'(\tau_0) = -H_{[f,[g,[f,g]]}(\Gamma(\tau_0))$. Differentiating (3.11) with respect to τ, we obtain also that $H_{[f,[f,[f,g]]]}(\Gamma(\tau)) = 0$. Combining these two facts, we obtain that matrix (2.10) has the form

$$\begin{pmatrix} P'(\tau_0) & H_{[g,[g,[f,g]]]}(\Gamma(\tau_0)) \\ 0 & -P'(\tau_0) \end{pmatrix}$$

at the point $\Gamma(\tau_0)$. Since this matrix is not degenerated in the hyperbolic case, we conclude that

$$P'(\tau_0) \neq 0. \tag{3.13}$$

Thus, if τ_0 is inside $[0,T]$, then $P(\tau)$ changes sign and by arguments similar to the classical calculus of variation (see [1], Appendix 2) the second variation of Γ is neither nonnegative nor nonpositive in this case. Using the mentioned above necessary condition for rigidity of an abnormal extremal of corank 1, we have

Proposition 3.2. *If an abnormal extremal γ has corank 1 and its abnormal lift Γ contains a hyperbolic point inside, then γ is not locally rigid.*

The rest of the paper is devoted to the case, where one endpoint (for example, $\Gamma(0)$) is hyperbolic and all other points of Γ are regular. We call it *hyperbolic single-endpoint case*. In this case $P(0) = 0$ and we can suppose that $P(\tau) > 0$ for $\tau > 0$ (i.e., Legendre, but not strong Legendre condition is fulfilled). Note that in the hyperbolic single-endpoint case we cannot obtain immediately the estimates, similar to (3.12). The following notion plays the key role in our work:

Definition 3.2. A quasi-normal basis (f, g) of the distribution D with respect to the abnormal extremal γ is called diagonalizing with respect to the abnormal lift $\Gamma(\tau) = (\gamma(\tau), \lambda(\tau))$, if

$$\lambda(0)\cdot\Big[Z_s\,[f,g], Z_\tau\,[f,g]\Big](q_0) \equiv 0, \quad 0 \le \tau, s \le T. \tag{3.14}$$

The right-hand side of (3.14) is actually the kernel $K(\tau, s)$ of (3.8). Thus, if the basis (f, g) is diagonalizing w.r.t. Γ, then the corresponding second variation has the following simple form:

$$2\lambda(T)\cdot F''_{\tilde{v}(\cdot)}v(\cdot) = \int_0^T P(\tau)w^2(\tau)\,d\tau. \tag{3.15}$$

In this case we say that the second variation of Γ w.r.t. the basis (f, g) has a *diagonal form.*

Theorem 3.1. *Suppose that an abnormal biextremal $\Gamma(\tau) = (\gamma(\tau), \lambda(\tau))$ is either regular or has one hyperbolic point as an endpoint. If there exists a diagonalizing basis w.r.t. Γ, then the corresponding abnormal extremal γ is a rigid curve.*

For the regular case the existence of diagonalizing basis implies inequality (3.12). Therefore, for this case the previous theorem follows from [1] (Theorem 4.8). For the hyperbolic single-endpoint case by (3.13) one have another estimate from below for the second variation in the diagonal form:

$$\lambda(T)\cdot F''_{\tilde{v}(\cdot)}v(\cdot) > \beta\int_0^T \tau\, w^2(\tau)\,d\tau \tag{3.16}$$

for some positive β. To prove Theorem 3.1 in this case, we modify the proof of the mentioned theorem from [1]. This modification is presented in Appendix B.

Now we want to investigate the question of the existence of diagonalizing basis. First, for the case $n = 3$, this question has very simple answer:

Theorem 3.2. *If $n = 3$, the diagonalizing basis exists both for regular abnormal biextremals and for abnormal biextremals with one hyperbolic point as an endpoint.*

Proof. First, we prove that under the assumption of the theorem there exists a quasi-normal basis w.r.t. γ. Indeed, in our case the Martinet set S is a smooth surface. In a neighborhood U of γ one can choose the coordinates (x_1, x_2, x_3) such that $S = \{x_2 = 0\}$ and $\gamma(t) = (t, 0, 0)$. Let (f, g) be the basis of D in U of the form $f = \dfrac{\partial}{\partial x_1}$, $g = \hat{g}_2 \dfrac{\partial}{\partial x_2} + \hat{g}_3 \dfrac{\partial}{\partial x_3}$. In the regular case $D(\gamma(\tau))$ is transversal to $T_{\gamma(\tau)}S$. Therefore, $\hat{g}_2(\gamma(\tau)) \neq 0$. Dividing g by $\hat{g}_2$, we get a quasi-normal basis.

In the hyperbolic one endpoint case $D(q_0) = T_{q_0}S$, hence $g_2(q_0) = 0$. Let $\alpha(\tau)$ be the angle between $D(\gamma(\tau))$ and $T_{\gamma(\tau)}S$ in the coordinate (x_1, x_2, x_3) such that the function $\alpha(\tau)$ is continuous and $\alpha(0) = 0$. Since $D(\gamma(\tau))$ is transversal to $T_{\gamma(\tau)}S$ for any $\tau > 0$, there exists a number $0 < \beta < \pi$ such that $|\alpha(\tau)| < \beta$. Take the surface $S' = \{\operatorname{ctg} \beta \, x_2 = x_3\}$. Then by the construction $D(\gamma(\tau))$ is transversal to $T_{\gamma(\tau)}S'$ for any τ. Defining near γ the coordinates (x_1', x_2', x_3') such that $S' = \{x_2' = 0\}$, one can find a quasi-normal basis w.r.t. γ by the same argument as in the regular case.

To complete the proof, we show that any quasi-normal basis w.r.t. γ is diagonalizing w.r.t. Γ. In fact, suppose that the basis (f, g) has the form $f = \dfrac{\partial}{\partial x_1}$, $g = \dfrac{\partial}{\partial x_2} + g_3 \dfrac{\partial}{\partial x_3}$. Then $[f, g] = \dfrac{\partial}{\partial x_1} g_3(\gamma(\tau)) \dfrac{\partial}{\partial x_3}$. On the other hand, from abnormality of γ it follows that, in the case of $n = 3$, $[f, g](\gamma(\tau))$ belongs to $\operatorname{span}(f(\gamma(\tau)), g(\gamma(\tau)))$ for any τ. Therefore, $[f, g](\gamma(\tau)) \equiv 0$. This implies (3.14). Thus, the basis (f, g) is diagonalizing w.r.t. Γ. $\square$

4. Jacobi curve

For $n \geq 3$ we will relate the existence of a diagonalizing basis (both in regular and hyperbolic one endpoint cases) with the properties of a special curve of Lagrangian subspaces in appropriate symplectic space, so-called *Jacobi curve*, which will be associated with any abnormal biextremal.

Now we want to describe a construction of Jacobi curve. For any point $\Gamma(\tau)$ let $\hat{V}_\tau$ be a vertical subspace of $T_{\Gamma(\tau)}(T^*M)$, i.e., $\operatorname{Ker} \pi_*(\Gamma(\tau))$. Let

$$\hat{L}_\tau = \Big(\hat{V}_\tau + \vec{H}_D(\Gamma(\tau))\Big) \bigcap T_{\Gamma(\tau)}\big((D^2)^\perp\big). \tag{4.1}$$

Note that actually

$$\hat{L}_\tau = \big\{e \in T_{\Gamma(\tau)}(D^2)^\perp : \pi_* e \in D(\pi\Gamma(\tau))\big\}. \tag{4.2}$$

By the construction

$$\sigma(\Gamma(\tau))|_{\hat{L}_\tau} \equiv 0, \tag{4.3}$$

$$\dim \hat{V}_\tau \bigcap T_{\Gamma(\tau)}(D^\perp) = n - 2, \quad \dim \vec{H}_D(\Gamma(\tau)) = 2. \tag{4.4}$$

Now we consider separately regular and hyperbolic single-endpoint cases.

4.1. Regular case. If the point $\Gamma(\tau)$ is regular, there exists a vector $e \in \vec{H}_D(\Gamma(\tau)$ such that

$$T_{\Gamma(\tau)}D^\perp = T_{\Gamma(\tau)}(D^2)^\perp + \operatorname{span}(e). \tag{4.5}$$

From (4.4) and (4.5) it follows that

$$\dim \hat{L}_\tau = n - 1. \tag{4.6}$$

Note that if $\Gamma(\tau) = (\gamma(\tau), \lambda(\tau))$ is an abnormal lift of γ, then $\Gamma_\alpha(\tau) = (\gamma(\tau), \alpha\lambda(\tau))$ is also an abnormal lift of γ for any number $\alpha \neq 0$. One can choose a characteristic field Ab in a neighborhood of Γ such that all curves Γ_α that are close to Γ are integral curves of Ab. Let h_t be a flow, generated by Ab. We translate subspaces $\hat{L}_\tau$ by the flow h_τ to the beginning of Γ, namely, we define subspaces $L'_\tau = h_{\tau*}^{-1}\hat{L}_\tau \subset T_{\Gamma(0)}(D^2)^\perp$. Given the point $Q = (q, \lambda)$ in T^*M, denote by $\vec{Q} \in T_Q(T^*M)$ the tangent vector to the curve $\alpha \to (q, \alpha\lambda)$ at the point $\alpha = 1$. By the construction $\vec{\Gamma}(0)$ belongs to L'_τ for any τ. Denote by $(\vec{\Gamma}(0))^\angle = \{v \in T_{\Gamma(0)}(D^2)^\perp : \sigma_2(v, \vec{\Gamma}(0)) = 0\}$. Let $W = (\vec{\Gamma}(0))^\angle/\operatorname{span}(\vec{\Gamma}(0), \dot{\Gamma}(0))$ and let p be the projection from $(\vec{\Gamma}(0))^\angle$ to W. Recall that $\operatorname{Ker}\sigma_2(\Gamma(\tau)) = \operatorname{span}(\dot{\Gamma}(\tau))$. Thus, the antisymmetric form $\Lambda(v_1, v_2) = \sigma_2(p^{-1}v_1, p^{-1}v_2)$ is well defined and nondegenerate on W. Hence the pair (W, Λ) is a symplectic space with $\dim W = 2(n-3)$. Define

$$L_\tau = p(L'_\tau). \tag{4.7}$$

Using (4.6) and the fact that $\operatorname{span}(\vec{\Gamma}(0), \dot{\Gamma}(0)) \subset L'_\tau$ for any τ, we obtain that $\dim L_\tau = n - 3$. It follows from (4.3) that $\Lambda|_{L_\tau} \equiv 0$. Therefore the curve $\tau \to L_\tau$ is a curve of Lagrangian subspaces of W.

Definition 4.1. The curve $\tau \to L_\tau$ is called Jacobi curve of the regular abnormal biextremals Γ.

Remark 4.1. In the construction of the Jacobi curve of regular biextremals we have a freedom to choose the characteristic field Ab. Since we make factorization by $\operatorname{span}(\dot{\Gamma}(0))$ on some step of the construction, the Jacobi curve finally does not depend on this choice.

4.2. Hyperbolic single-endpoint case. For the sake of the definiteness, suppose that the characteristic field Ab of D is chosen such that $\Gamma(\tau)$ geometrically coincide with a stable invariant curve of $\Gamma(0)$. All points of some neighborhood of $\Gamma(0)$ in $(D^3)^\perp$ are hyperbolic. Denote by Sep the union of stable invariant curves of all these points. Using Proposition 2.4, we obtain that Sep is a smooth manifold in the neighborhood of Γ.

Though the restriction of the characteristic vector field Ab to Sep has stationary points, we can get rid of them by taking another vector field $\tilde{A}b$ on Sep without stationary points, but such that $\tilde{A}b$ and Ab define the same field of directions on Sep . As in the regular case, we can choose the field $\tilde{A}b$ in a neighborhood of Γ such that all curves Γ_α close to Γ are integral curves of $\tilde{A}b$. To construct Jacobi curve in the hyperbolic one endpoint case we want to use a scheme similar to the regular case, taking the manifold Sep instead of $(D^2)^\perp$ and the field $\tilde{A}b$ instead of Ab. The flow on Sep generated by $\tilde{A}b$ is also denoted by h_t.

Now we want to divide the points of $(D^3)^\perp$ into two types: the point $Q \in (D^3)^\perp$ is called *of the type* $(2,3)$, if $\dim D^2(\pi Q) = 3$, and *of the type* $(2,2)$, if $\dim D^2(\pi Q) = 2$.

Suppose that $\Gamma(0)$ is the point of the type $(2,3)$, then $T_{\Gamma(0)}(D^\perp) = T_{\Gamma(0)}(D^2)^\perp + \operatorname{span}(\vec{\Gamma}(0))$. Now if we recall (4.4), we get as in the regular case that

$$\dim \hat{L}_0 = n - 1. \tag{4.8}$$

If $\Gamma(0)$ is the point of the type $(2,2)$, then the space $\big(V_0 \cap T_{\Gamma(0)}(D)^\perp + \vec{H}_D(\Gamma(0))\big)$ is tangent to $(D^2)^\perp$. Hence

$$\dim \hat{L}_0 = n. \tag{4.9}$$

Let

$$\tilde{L}_\tau = (\hat{V}_\tau + \vec{H}_D(\Gamma(\tau))) \cap T_{\Gamma(\tau)}(\text{Sep}\,) = \hat{L}_\tau \cap T_{\Gamma(\tau)}(\text{Sep}\,)$$

There are 2 possibilities:

(1) $\hat{L}_\tau$ is transversal to $T_{\Gamma(\tau)}(\text{Sep}\,)$ in $T_{\Gamma(\tau)}(D^2)^\perp$,

(2) $\hat{L}_\tau \subset T_{\Gamma(\tau)}(\text{Sep}\,)$.

Definition 4.2. If $\hat{L}_\tau$ satisfies condition (1), we say that the point $\Gamma(\tau)$ of the abnormal biextremal satisfies the *transversality condition*.

Note that $\Gamma(0)$ satisfies the transversality condition. Actually, the second separatrix of $\Gamma(0)$ is transversal to Sep . On the other hand, the tangent vector to this separatrix at the point $\Gamma(0)$ is contained in $\vec{H}_D(\Gamma(0) \subset \hat{L}_0$. Thus, L_0 is transversal to $T_{\Gamma(0)}\text{Sep}$. Let $\Gamma(0)$ be of the type $(2,3)$. Taking for any point $\Gamma(\tau)$ the subspace $\tilde{L}_\tau$, we obtain a smooth $(n-1)$-dimensional bundle over the curve Γ (see (4.6) and (4.8)). This proves that:

Proposition 4.1. *If* $\Gamma(0)$ *is the hyperbolic point of the type* $(2,3)$, *then* $\Gamma(\tau)$ *satisfies the transversality condition for sufficiently small* τ.

Remark 4.2. For $\Gamma(0)$ of the type $(2,2)$ the previous proposition in general is not true, since we have the jump of the dimension of the spaces $\hat{L}_\tau$ at $\tau = 0$ (see (4.9)). For instance, it is easy to prove that if $n = 3$, $\Gamma(\tau)$ does not satisfy the transversality condition for any $\tau > 0$.

Denote by $\sigma_3 = \sigma|_{\mathrm{Sep}}$ and let $K_\tau = \operatorname{Ker} \sigma_3(\Gamma(\tau))$. Since $\dim \mathrm{Sep} = 2n-4$ and $\dot{\Gamma}(\tau) \in K_\tau$, then $\dim K_\tau \geq 2$.

Lemma 4.1. $\dim K_\tau = 2$ *for all* $\tau \in [0, T]$.

Proof. For $\tau > 0$ it follows immediately from the fact that $\operatorname{Ker} \sigma_2(\Gamma(\tau)) = \operatorname{span}(\dot{\Gamma}(\tau))$. For $\tau = 0$ we use the fact that the flow h_t preserves the form σ_3. □

Now let us prove the following two propositions:

Proposition 4.2. *For* $\tau > 0$, $\hat{L}_\tau \cap K_\tau = \operatorname{span}(\dot{\Gamma}(\tau))$ *iff the point* $\Gamma(\tau)$ *satisfies the transversality condition.*

Proposition 4.3. $\hat{L}_0 \cap K_0 = \operatorname{span}(\dot{\Gamma}(0))$ *iff the point* $\Gamma(0)$ *is a hyperbolic point of the type* $(2,3)$.

Proof of Proposition 4.2. Sufficiency. Suppose that the point $\Gamma(\tau)$ satisfies the transversality condition. Let a vector $e \in \hat{L}_\tau \cap K_\tau$. From the transversality condition there exists $e_1 \in \hat{L}_\tau$ such that $T_{\Gamma(\tau)}(D^2)^\perp = T_{\Gamma(\tau)}(\mathrm{Sep}) + \operatorname{span}(e_1)$. From (4.3) it follows that $\sigma_2(e, e_1) = 0$, hence $e \in \operatorname{Ker} \sigma_2$. Recall that for $\tau > 0$ the point $\Gamma(\tau)$ is regular and $\operatorname{Ker} \sigma_2(\Gamma(\tau)) = \operatorname{span}(\dot{\Gamma}(\tau))$. This implies that $e \in \operatorname{span}(\dot{\Gamma}(\tau)) \Rightarrow \hat{L}_\tau \cap K_\tau = \operatorname{span}(\dot{\Gamma}(\tau))$.

Necessity. Taking the factorization of $T_{\Gamma(\tau)}(\mathrm{Sep})$ by K_τ, we get a symplectic space of dimension $\dim T_{\Gamma(\tau)}(\mathrm{Sep}) - \dim K_\tau = 2n - 6$. The image of $\tilde{L}_\tau$ under this factorization is an isotropic subspace of the obtained symplectic space. Thus, its dimension is not greater than $n - 3$. On the other hand, the dimension of this space is equal to $\dim \hat{L}_\tau \cap T_{\Gamma(\tau)}(\mathrm{Sep}) - \dim \hat{L}_\tau \cap K_\tau = \dim \hat{L}_\tau \cap T_{\Gamma(\tau)}(\mathrm{Sep}) - 1$. If $\hat{L}_\tau \cap K_\tau = \operatorname{span}(\dot{\Gamma}(\tau))$, we have $\dim \hat{L}_\tau \cap T_{\Gamma(\tau)}(\mathrm{Sep}) \leq n-2$. Combining this inequality and (4.6), we obtain that $\hat{L}_\tau$ is transversal to $T_{\Gamma(\tau)}(\mathrm{Sep})$ in $T_{\Gamma(\tau)}(D^2)^\perp$. □

Proof of Proposition 4.3. Sufficiency. Suppose that the point $\Gamma(0)$ is of the type $(2,3)$. Let a vector $e \in \hat{L}_\tau \cap K_\tau$. Since $\Gamma(0)$ satisfies the transversality condition, then $e \in \operatorname{Ker} \sigma_2$ (see the previous proof). Note that $T_{\Gamma(0)}(D^\perp) = T_{\Gamma(0)}(D^2)^\perp + \operatorname{span}(\vec{\Gamma}(0))$ and $(\vec{\Gamma}(0) \rfloor \sigma)|_{\hat{L}_\tau} = 0$. Therefore, $e \in \operatorname{Ker} \sigma_1$. Using (2.5), we have $e \in \vec{H}_D(\Gamma(0))$. Since $\vec{H}_D(\Gamma(0)) \cap T_{\Gamma(0)}\mathrm{Sep} = \operatorname{span}(\dot{\Gamma}(0))$, we obtain that $e \in \dot{\Gamma}(0) \Rightarrow \hat{L}_0 \cap K_0 = \operatorname{span}(\dot{\Gamma}(0))$.

Necessity. If $\hat{L}_0 \cap K_0 = \operatorname{span}(\dot{\Gamma}(0))$, then, as in the previous proof, $\dim \hat{L}_0 \cap T_{\Gamma(0)}(\mathrm{Sep}) \leq n - 2$. This implies that $\dim \hat{L}_0 \leq n - 1$. But, in the case of $\Gamma(0)$ of the type $(2,2)$, $\dim \hat{L}_0 = n$. Therefore, the point $\Gamma(0)$ is necessarily of the type $(2,3)$. $\square$

As in the regular case we can translate $\tilde{L}_\tau$ to the beginning of Γ: $L'_\tau = h_{\tau*}^{-1}\tilde{L}_\tau$. Denote by $(\vec{\Gamma}(0))^\angle = \{v \in T_{\Gamma(0)}(\mathrm{Sep}) : \sigma_2(v, \vec{\Gamma}(0)) = 0\}$. Let $W = (\vec{\Gamma}(0))^\angle/\mathrm{span}(\vec{\Gamma}(0), K_0)$ and let p be the projection from $(\vec{\Gamma}(0))^\angle$ to W. The antisymmetric form $\Lambda(v_1, v_2) = \sigma_2(p^{-1}v_1, p^{-1}v_2)$ is well defined and nondegenerate on W. Hence the pair (W, Λ) is a symplectic space. If $\vec{\Gamma}(0)$ does not belong to K_0, then

$$\dim W = 2(n-4). \tag{4.10}$$

If $\vec{\Gamma}(0)$ belongs to K_0, then

$$\dim W = 2(n-3). \tag{4.11}$$

Lemma 4.2. *If $\vec{\Gamma}(0)$ belongs to K_0, then the point $\Gamma(0)$ is of the type $(2,2)$ and $\Gamma(\tau)$ does not satisfy the transversality condition for any $\tau > 0$.*

Proof. By the construction, $\vec{\Gamma}(0) \in L'_\tau$ for all $\tau \Rightarrow K_0 \subset L'_\tau \Rightarrow K_\tau \subset \tilde{L}_\tau$. Now using Propositions 4.2 and 4.3, we obtain that $\Gamma(0)$ is of the type $(2,2)$ and $\Gamma(\tau)$ does not satisfy the transversality condition for any $\tau > 0$. $\square$

Define

$$L_\tau = p(L'_\tau). \tag{4.12}$$

The spaces L_τ are obviously isotropic subspaces of W. Let us prove

Proposition 4.4. *The subspace L_τ is a Lagrangian subspace of W for any τ.*

Proof. Let us check all possibilities: (1) Suppose that $\vec{\Gamma}(0)$ belongs to K_0. We need to prove that $\dim L_\tau = n-3$ (see (4.11)).

Using Lemma 4.2, formula (4.9), and the fact that $\hat{L}_0$ is transversal to $T_{\Gamma(\tau)}\mathrm{Sep}$, we obtain that

$$\dim L'_0 = n-1. \tag{4.13}$$

Consequently, $\dim L_0 = n-3$.

If $\tau > 0$, then $K_\tau \subset \hat{L}_\tau$ and $\tilde{L}_\tau = \hat{L}_\tau$. By (4.6), we have $\dim L_\tau = n-3$.

(2) Suppose that $\vec{\Gamma}(0)$ does not belong to K_0, then $\dim \mathrm{span}(\vec{\Gamma}(0), K_0) = 3$. We need to prove that $\dim L_\tau = n-4$ (see (4.10)).

Let $\Gamma(0)$ be of the type $(2,2)$. Then $\mathrm{span}(\vec{\Gamma}(0), K_0) \subset L'_0$ and by (4.13), $\dim L_0 = n-4$.

Let $\Gamma(0)$ be of the type $(2,3)$. Then by the assumption

$$\dim \mathrm{span}(\vec{\Gamma}(0), K_0) \cap L' = 2$$

and by formula (4.8) $\dim L'_0 = n-2 \Rightarrow \dim L_0 = n-4$.

Let $\tau > 0$ and let the point $\Gamma(\tau)$ satisfy the transversality condition, then by (4.6), it follows that $\dim L'_\tau = n-2$, and from Proposition 4.2 we obtain that $\dim L'_\tau \cap \operatorname{span}(\vec{\Gamma}(0), K_0) = 2$. Therefore, $\dim L_\tau = n-4$.

Let $\tau > 0$ and let the point $\Gamma(\tau)$ do not satisfy transversality condition. Then by (4.6), it follows that $\dim L'_\tau = n-1$, and from Proposition 4.2 we obtain that $\dim L'_\tau \cap \operatorname{span}(\vec{\Gamma}(0), K_0) = 3$. Therefore, $\dim L_\tau = n-4$. $\square$

Definition 4.3. The curve $\tau \to L_\tau$ is called Jacobi curve of the abnormal biextremals Γ with one hyperbolic endpoint.

Remark 4.3. By the same reason, as in the regular case (see Remark 4.1), the Jacobi curve of biextremals with one hyperbolic endpoint does not depend on a freedom in the choice of the field $\tilde{Ab}$ on Sep .

Remark 4.4. In the regular case, the Jacobi curve is a continuous curve of Lagrangian Grassmannian of W. In the hyperbolic one endpoint case, the Jacobi curve can be discontinuous at $\tau = 0$, if $\Gamma(0)$ is of the type $(2,2)$, or at $\tau > 0$, if $\Gamma(\tau)$ does not satisfy the transversality condition, since the spaces $\hat{L}_\tau$ and $\hat{L}_\tau \cap K_\tau$ have jumps of dimension at these points.

Recall that the curve of Lagrangian subspaces is called simple, if there exists a Lagrangian subspace transversal to any subspace of the curve.

Theorem 4.1. (*Sufficient condition for the existence of a diagonalizing basis.*) *If the Jacobi curve of the abnormal biextremal* Γ *is simple then*

(1) *in the regular case the diagonalizing basis w.r.t..* Γ *exists;*

(2) *in the case of hyperbolic one endpoint of the type* (2,3) ($n > 3$) *the same holds provided that any point of* Γ *satisfies the transversality condition.*

Proof. To give the proof in the regular and nonregular cases simultaneously, let us introduce some common notations. Denote by E the space $T_{\Gamma(0)}(D^2)^\perp$ in the regular case and the space $T_{\Gamma(0)}\mathrm{Sep}$ in the nonregular case. In the regular case we also write K_τ instead of $\operatorname{span}(\dot{\Gamma}(\tau))$, that is the kernel of the form $\sigma_2(\Gamma(\tau))$.

Let Δ be the Lagrangian subspace in W that is transversal to any L_τ and let $p^{-1}\Delta$ be its preimage under the projection of $(\vec{\Gamma}(0))^\angle$ to W. Take a vector e transversal to $(\vec{\Gamma}(0))^\angle$ in E and denote by $e^\angle = \{v \in E : \sigma(v,e) = 0\}$. Let $\tilde{\Delta} = p^{-1}\Delta \cap e^\angle + \operatorname{span}(e)$. Since $\vec{\Gamma}(0) \notin K_0$ in the considered cases, we have

$$\tilde{\Delta} \cap \operatorname{span}(\vec{\Gamma}(0), K_0) = K_0. \tag{4.14}$$

By the construction, $\dim \tilde{\Delta} = n-1$ and $\sigma(\Gamma(0))|_{\tilde{\Delta}} = 0$. Translating the space $\tilde{\Delta}$ by the flow h_τ, we obtain the spaces $\tilde{\Delta}_\tau = h_{\tau *}\tilde{\Delta}$.

Lemma 4.3. $\tilde{\Delta}_\tau \cap \hat{V}_\tau = 0$.

Proof. By the construction $\tilde{\Delta} \cap L'_\tau \subset \operatorname{span}(\vec{\Gamma}(0), K_0)$. From (4.14) it follows that $\tilde{\Delta} \cap L'_\tau \subset K_0 \cap L'_\tau$. Translating this relation by the flow h_τ and using in the nonregular case Propositions 4.2 and 4.3, we obtain

$$\tilde{\Delta}_\tau \cap \hat{L}_\tau \subset K_\tau \cap \hat{L}_\tau = \operatorname{span}(\dot{\Gamma}(\tau)). \tag{4.15}$$

Therefore, $\tilde{\Delta}_\tau \cap \hat{V}_\tau \subset \operatorname{span}(\dot{\Gamma}(\tau)) \cap \hat{V}_\tau = 0$. □

Let S be a smooth submanifold of E of dimension $n-2$, such that $T_{\Gamma(0)}S \subset \tilde{\Delta}$ and S is transversal to Γ at the point $\Gamma(0)$. Consider the set Σ, obtained by translation of S along the flow h_t:

$$\Sigma = \bigcup_{0 \le \tau \le T} h_\tau(S),$$

Σ is submanifold in the neighborhood of Γ and $T_{\Gamma(\tau)}\Sigma = \tilde{\Delta}_\tau$.

Let $N = \pi\Sigma$ be the projection of Σ to the base M and let $\pi_*\Delta\tau = \delta_\tau$. Using Lemma 4.3 and implicit function theorem, we obtain that the projection π has an inverse $\Psi : N \to \Sigma$. One can define a differential 1-form λ on N such that $\Psi(q) = (q, \lambda(q))\ \forall q \in N$ (the manifold Σ is actually the graph of the form λ). By the construction, the flow h_t can be restricted to Σ. The projection $\pi : \Sigma \to N$ defines the flow $P_t = \pi h_\tau$ of abnormal extremals on N. One can introduce coordinates $(x_1, \dots, x_n)$ in the neighborhood of γ on M such that $N = \{x_2 = 0\}$, the field $\dfrac{\partial}{\partial x_1}$ is tangent to D and the flow P_t is generated by the field $\dfrac{\partial}{\partial x_1}$. In these coordinates

$$\delta_\tau = \operatorname{span}\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_3}, \dots, \frac{\partial}{\partial x_n}\right) \tag{4.16}$$

(i.e., without $\dfrac{\partial}{\partial x_2}$).

Note that $d\lambda = \Psi^*(\sigma)$. This yields that

$$d\lambda|_{\delta_\tau} (= \sigma|_{\tilde{\Delta}_\tau}) = 0 \tag{4.17}$$

for all $\tau \in [0, T]$.

The basis of the distribution D can be chosen in the form (3.2):

$$f = \frac{\partial}{\partial x_1},$$
$$\hat{g} = \sum_{i=2}^{n} \hat{\phi}_i \frac{\partial}{\partial x_i}.$$

Lemma 4.4. $\hat{\phi}_2(\gamma(\tau)) \neq 0 \ \forall \tau \in [0, T]$.

Proof. Assume that $\hat{\phi}(\tau_0) = 0$ for some τ_0, then $\hat{g}(\gamma(\tau_0)) \in \delta_{\tau_0}$. It implies the existence of the vector $\mu \in \tilde{\Delta}_{\tau_0}$ such that

$$\pi_* \mu = \hat{g}(\gamma(\tau_0)). \tag{4.18}$$

Therefore from (4.2) $\mu \in \hat{L}_{\tau_0}$. Thus, $\mu \in \tilde{\Delta}_{\tau_0} \cap \hat{L}_{\tau_0}$. Using (4.15), we obtain that $\mu \in \operatorname{span}(\dot{\Gamma}(\tau_0))$. Consequently, $\pi_* \mu \in \operatorname{span}(f(\gamma(\tau_0))$ that contradicts to (4.18). $\square$

Thus, $\hat{\phi}_2(\gamma(\tau)) \neq 0$. Dividing $\hat{g}$ by $\hat{\phi}_2(\gamma(\tau))$, we obtain a quasi-normal basis:

$$f = \frac{\partial}{\partial x_1},$$
$$g = \frac{\partial}{\partial x_2} + \sum_{i=3}^{n} \phi_i \frac{\partial}{\partial x_i}.$$

It is clear that $[f, g](\gamma(\tau)) \in \operatorname{span}\left(\left\{\frac{\partial}{\partial x_i}\right\}_{i=3}^{n}\right) \subset \delta_\tau$ for all τ. Combining this and the fact that $Z_\tau[f, g](\gamma(0)) = P_{\tau*}^{-1}[f, g](\gamma(\tau))$, we obtain

$$Z_\tau[f, g](\gamma(0)) \in \delta_0. \tag{4.19}$$

This formula and (4.17) imply that

$$d\lambda(\gamma(0))(Z_\tau[f, g], Z_s[f, g]) \equiv 0 \quad \forall 0 \leq \tau, s \leq T. \tag{4.20}$$

By the construction, the curve $\tau \to (P_\tau q, \lambda(P_\tau q))$ is an abnormal biextremal for any $Q \in N$. Using (3.6), one can easily obtain that

$$\lambda(q)(Z_\tau[f, g]) = \lambda(P_\tau q))([f, g])$$

for any $q \in N$. Recall that $\Sigma \subset (D^2)^\perp$, therefore $\lambda(q)([f, g]) = 0$ for any $q \in N$. Consequently,

$$\lambda(q)(Z_\tau[f, g]) = 0. \tag{4.21}$$

Formulas (4.19) and (4.21) imply that the derivative of the function $\lambda(q)(Z_\tau[f, g])$ at the point $q = \gamma(0)$ in the direction $Z_s[f, g](\gamma(0))$ is equal to 0 for any τ and s. We denote this directional derivative by

$$Z_s[f, g](\gamma(0))\ \lambda(\gamma(0))(Z_\tau[f, g]).$$

Finally, using the well-known Cartan identity and (4.20), we obtain:

$$K(\tau,s) = \lambda(\gamma(0))\Big(\Big[Z_\tau[f,g], Z_s[f,g]\Big]\Big) = d\lambda(\gamma(0))(Z_s[f,g], Z_\tau[f,g]) + \\ + Z_s[f,g](\gamma(0))\ \lambda(\gamma(0))(Z_\tau[f,g]) - \\ - Z_\tau[f,g](\gamma(0))\ \lambda(\gamma(0))(Z_s[f,g]) = 0$$

for all $0 \le \tau,\ s \le T$. Thus, the basis (f,g) is diagonalizing w.r.t. Γ. $\square$

Now we formulate the main result of our paper. This result is an immediate corollary of Theorems 3.1, 3.2, and 4.1.

Theorem 4.2. *Let $\Gamma(\tau) = (\gamma(\tau), \lambda(\tau))$ be an abnormal biextremal. Then*

(a) *Let $n = 3$. If Γ is regular or has one hyperbolic point as an endpoint, then γ is rigid.*

(b) *Let $n > 3$. If the Jacobi curve of Γ is simple, then*
   (1) *in the regular case γ is rigid,*
   (2) *in the case of hyperbolic one endpoint of the type $(2,3)$ γ is rigid provided that any point of Γ satisfies the transversality condition.*

Proposition 4.1, Remark 4.4, and the fact that any sufficiently small part of a continuous curve in Lagrangian Grassmannian is simple imply the following result:

Corollary 4.1. *If $n > 3$ and $\Gamma(\tau) = (\gamma(\tau), \lambda(t))$ is an abnormal biextremal which is either regular or has one hyperbolic point of the type $(2,3)$ as an endpoint, then γ is locally rigid.*

Remark 4.5. In [15], in addition to part (a) of Theorem 4.2, we proved that for $n = 3$ if both endpoints of Γ are hyperbolic and all other points of Γ are regular, then abnormal extremal γ is rigid. We believe that this result can be also obtained by the methods of the present paper.

5. Appendix A

In this section we give a short list of notation and formulas from chronological calculus, developed in [5]. Geometrical objects, like C^∞ mappings and vector fields on the manifold M can be identified with certain linear operators, acting on $C^\infty(M)$. Namely, a C^∞ mapping P on M can be associated with the shift operator, given by the following rule: $P\phi(q) = \phi(Pq)$ for any $\phi \in C^\infty(M)$ and $q \in M$. Any point $q_0 \in M$ is associated with the constant mapping that maps whole M to q_0 and this mapping can be identified with the operator (or functional) $\phi \to \phi(q_0)$ on $C^\infty(M)$. This operator will be also denoted by q_0. A smooth vector field X on M is on one hand a smooth section of the tangent bundle TM and on the other hand it is a first order differential operator on M. Further, one can identify a nonautonomic ordinary differential equation

$$\dot{q} = X(q,t), \quad q(0) = q_0 \tag{5.1}$$

with the operator differential equation

$$\frac{d}{dt}P_t = P_t \circ X_t,\ P_0 = \mathrm{Id}, \tag{5.2}$$

where $X_t = X(\cdot, t)$, P_t is an operator associated with the flow, generated by (5.1), and the sign "$\circ$" means the composition of operators, $L_1 \circ L_2(\phi) = L_1(L_2(\phi))$. Operator P_t is called the right chronological exponent of X_t and denoted by $\overrightarrow{\exp} \int\limits_0^t X_\tau d\tau$.

Integration of (5.2) by the parameter gives us:

$$P_t = \mathrm{Id} + \int\limits_0^t P_{\tau_1} \circ X_{\tau_1}\, d\tau_1. \tag{5.3}$$

The advantage of operator equation (5.2) in comparison with (5.1) is the linearity of the operators P_t. Thus, one can iterate equation (5.3) arbitrary number of times. After m iterations we get the following Volterra expansion of the flow $\overrightarrow{\exp} \int_0^t X_\tau d\tau$:

$$\overrightarrow{\exp} \int\limits_0^t X_\tau d\tau =$$

$$= \mathrm{Id} + \sum_{i=1}^{m} \int\limits_0^t d\tau_1 \int\limits_0^{\tau_1} d\tau_2 \ldots \int\limits_0^{\tau_{i-1}} X_{\tau_i} \circ \ldots \circ X_{\tau_1} d\tau_i + R_{m+1}, \tag{5.4}$$

where the remainder R_{m+1} has the form:

$$R_{m+1} =$$

$$= \int\limits_0^t d\tau_1 \int\limits_0^{\tau_1} d\tau_2 \ldots \int\limits_0^{\tau_m} \Big(\overrightarrow{\exp} \int\limits_0^{\tau_{m+1}} X_{\tau_{m+2}}\, d\tau_{m+2}\Big) \circ X_{\tau_{m+1}} \circ \ldots \circ X_{\tau_1} d\tau_{m+1}. \tag{5.5}$$

It is natural to denote the flow, generated by time-independent vector field X by e^{tX}.

Let $\|\cdot\|_{s,K}$ be some seminorm that defines the topology of the uniform convergence on the compact set $K \subset M$ of all derivatives up to the order s. One can define the following family of seminorms on the set of all vector fields on M: $\|X\|_{s,K} = \sup\big\{\|X\phi\|_{s,K} : \|\phi\|_{s+1,K} = 1\big\}$. Using analog of

Gronwall lemma, one can obtain the following estimates of the remainder R_{m+1}: for any function $\phi \in C^\infty(M)$, compact set $K \subset M$, and integer nonnegative number s there exist constants C_1 and C_2 and a compact set K', $K \subset K'$, such that

$$\|R_{m+1}\phi\|_{s,K} \leq$$
$$\leq C_1 e^{C_2 \int_0^t \|X_\tau\|_{s,M} d\tau} \left(\int\limits_0^t \|X_\tau\|_{s+m,K'}) d\tau\right)^{m+1} \|\phi\|_{s+m+1,K'}. \quad (5.6)$$

Note that if $M = \mathbb{R}^n$ and $R = \int\limits_0^t \|X_\tau\|_{0,\mathbb{R}^n} d\tau < \infty$, then the set $O_R(K)$ of all points with a distance to K not greater than R can be taken as the set K' in (5.6).

Given a diffeomorphism P and a vector field X on M denote by $\operatorname{Ad} P$ and $\operatorname{ad} X$ the following two operators on the set of all vector fields on M: $\operatorname{Ad} PY = P \circ Y \circ P^{-1} = P_*^{-1} Y$ and $(\operatorname{ad} X)Y = [X, Y]$. Perturbation $\overrightarrow{\exp} \int\limits_0^t (X_\tau + Y_\tau)\, d\tau$ of the flow $\overrightarrow{\exp} \int\limits_0^t X_\tau\, d\tau$ can be represented as the composition of two flows such that one of them is the original flow $\overrightarrow{\exp} \int_0^t X_\tau d\tau$. Namely,

$$\overrightarrow{\exp} \int\limits_0^t (X_\tau + Y_\tau)\, d\tau = \overrightarrow{\exp} \int\limits_0^t \operatorname{Ad}\Big(\overrightarrow{\exp} \int\limits_0^\tau X_\xi\, d\xi\Big) Y_\tau d\tau \circ \overrightarrow{\exp} \int\limits_0^t X_\tau d\tau \quad (5.7)$$

or

$$\overrightarrow{\exp} \int\limits_0^t (X_\tau + Y_\tau)\, d\tau = \overrightarrow{\exp} \int\limits_0^t X_\tau\, d\tau \circ \overrightarrow{\exp} \int\limits_0^t \operatorname{Ad}\Big(\overrightarrow{\exp} \int\limits_t^\tau X_\xi\, d\xi\Big) Y_\tau\, d\tau. \quad (5.8)$$

Applying one of these formulas to the one-parameter perturbation

$$\overrightarrow{\exp} \int\limits_0^t (X_\tau + \varepsilon Y_\tau)\, d\tau$$

and expanding the chronological exponents with Y_τ to the Volterra expansion, one can obtain the Taylor expansion of the family in ε. Thus, formulas (5.7), (5.8), and expansion (5.4) give the method of finding first, second and etc. differentials of the endpoint mapping.

Finally, if $P_t = \overrightarrow{\exp} \int_0^t X_\tau\, d\tau$, then by a direct calculation:

$$\frac{d}{dt} \operatorname{Ad} P_t Y = \operatorname{Ad} P_t [X_t, Y] = \operatorname{Ad} P_t (\operatorname{ad} X_t) Y \quad (5.9)$$

or $\frac{d}{dt}\operatorname{Ad} P_t = \operatorname{Ad} P_t \operatorname{ad} X_t$. Just as for (5.2), we can write

$$\operatorname{Ad}\left(\overrightarrow{\exp}\int_0^t X_\tau\, d\tau\right) = \overrightarrow{\exp}\int_0^t \operatorname{ad} X_\tau\, d\tau.$$

For time independent flows X this relation has the form $\operatorname{Ad} e^{tX} = e^{t\operatorname{ad} X}$. This notation will be very useful in the sequel.

6. Appendix B

In this appendix we prove Theorem 3.1 in the hyperbolic one endpoint case. As was mentioned above, the proof is actually a modification of the proof of the analogous result for the regular case, established in [1] (Theorem 4.8). We use the following sufficient condition for isolated points on the critical level set (see [1], [3]):

Theorem 6.1. *Let a Banach space X is densely imbedded into a Hilbert space H. Let a mapping $\mathcal{F}: X \to \mathbb{R}^m$ be Frechet differentiable at a point $\hat{x} \in X$, which is critical point of $\mathcal{F}$, i.e., $\lambda \cdot \mathcal{F}' = 0$ for some nonzero $\lambda \in \mathbb{R}^m$. If:*

$$(1)\ \|\mathcal{F}(\hat{x}+x) - \mathcal{F}(\hat{x}) - \mathcal{F}'(\hat{x})x\| = o(1)\|x\|_H, \quad \text{as} \quad \|x\|_X \to 0; \quad (6.1)$$

(2) *the function $\lambda \cdot \mathcal{F}$ admits Taylor expansion at $\hat{x}$ of the form*

$$\lambda \cdot \mathcal{F}(\hat{x}+x) - \lambda \cdot \mathcal{F}(x) = \frac{1}{2}\lambda \cdot \mathcal{F}''(\hat{x})(x,x) + o(1)\|x\|_H^2, \quad \text{as} \quad \|x\|_X \to 0; \quad (6.2)$$

(3) *the quadratic form $\lambda \cdot \mathcal{F}''(\hat{x})(x,x)$ admits a continuous extension from X onto H and is H-positive definite on $\operatorname{Ker} \mathcal{F}'(\hat{x})$, i.e., for some $\beta > 0$*

$$\lambda \cdot \mathcal{F}''(\hat{x})(\xi,\xi) \geq \beta\|\xi\|_H^2 \quad \forall \xi \in \operatorname{Ker} \mathcal{F}'(\hat{x}), \quad (6.3)$$

then $\hat{x}$ is an isolated point on the level set $\mathcal{F}^{-1}(\mathcal{F}(\hat{x}))$ in X.

Proof of Theorem 3.1. Suppose that (f,g) is a diagonalizing basis in a neighborhood of the curve γ and F is the corresponding endpoint mapping. Without loss of the generality it can be assumed that the image of F lies in some neighborhood of γ. Coordinatizing this neighborhood, we can suppose that the mapping F is actually into $\mathbb{R}^n$. We will verify the conditions of the previous theorem for the endpoint mapping F, $X = L_\infty[0,T]$ and the Hilbert space H that we want to describe now.

The space $\operatorname{Ker} F'_{\hat{v}(\cdot)}$ is a subspace of finite codimension in $L_\infty[0,T]$. Let the space Z be the complement of $\operatorname{Ker} F'_{\hat{v}(\cdot)}$ in $L_\infty[0,T]$. Any $v(\cdot) \in L_\infty[0,T]$

can be uniquely represented in the form $v(\cdot) = v_1(\cdot) + v_2(\cdot)$, where $v_1(\cdot) \in \operatorname{Ker} F'_{\hat{v}(\cdot)}$ and $v_2(\cdot) \in Z$. Let us define the following two norms:

$$\|v(\cdot)\|_1 = \left(w^2(T) + \int_0^T \tau w^2(\tau)\, d\tau \right)^{\frac{1}{2}}, \tag{6.4}$$

$$\|v(\cdot)\|_2 = \left(\|v_1(\cdot)\|_1^2 + \|v_2(\cdot)\|_1^2 \right)^{\frac{1}{2}}, \tag{6.5}$$

where $w(\cdot) = \int_0^{\cdot} v(\tau) d\tau$. Note that the norm $\|\cdot\|_2$ is at least not weaker, than $\|\cdot\|_1$. In general, these norms are not equivalent, since the first differential $F'_{\hat{v}(\cdot)}$ in the general case is not continuous in the norm $\|\cdot\|_1$.

We claim that the completion of $L_\infty[0,T]$ in the norm $\|\cdot\|_2$ can be taken as the space H. Further, we write $\|\cdot\|_H$ instead of $\|\cdot\|_2$ and denote by H_1 the completion of $L_\infty[0,T]$ in the norm $\|\cdot\|_1$.

First note that inequality (6.3) holds, since it is equivalent to (3.16). Since all estimates that we need are given in terms of the primitive $w(\cdot)$ of the control $v(\cdot)$, it is convenient to write the endpoint mapping F in terms of $w(\cdot)$ instead of $v(\cdot)$. This can be done with the help of so-called integration by parts formula for a chronological exponential (see [13]). We derive:

$$\begin{aligned} F(v(\cdot)) &= q_0 \circ \overrightarrow{\exp} \int_0^T \big(f + v(\tau) g\big)\, d\tau = q_1 \circ \overrightarrow{\exp} \int_0^T Y_\tau g v(\tau)\, d\tau = \\ &= q_1 \circ \overrightarrow{\exp} \int_0^T \left(- \int_0^1 e^{(1-\xi)\operatorname{ad}(Y_\tau g w(\tau))} \dot{Y}_\tau g w(\tau)\, d\xi \right) d\tau \circ e^{g w(T)}, \end{aligned}$$

where $Y_\tau = e^{(\tau - t)\operatorname{ad} f}$. Expanding all chronological and ordinary exponentials in the last formula, one can obtain the following expansion:

$$F(v(\cdot)) = q_1 + \underbrace{q_1 \circ \left(g\, w(T) - \int_0^T \dot{Y}_\tau g\, w(\tau) d\tau \right)}_{F'_{\hat{v}(\cdot)} v(\cdot)} +$$

$$\underbrace{+\, q_1 \circ \left(\frac{1}{2} \int_0^T [\dot{Y}_\tau g, Y_\tau g] w^2(\tau)\, d\tau + \right.}_{F''_{\hat{v}(\cdot)} v(\cdot) +}$$

$$\underbrace{+\int_0^T d\tau_1 w(\tau_1)\int_0^{\tau_1}\dot{Y}_{\tau_2}g\circ\dot{Y}_{\tau_1}gw(\tau_2)\,d\tau_2-}_{F''_{\hat{v}(\cdot)}v(\cdot)}$$

$$\underbrace{-\int_0^T \dot{Y}_\tau g\,w(\tau)d\tau\circ gw(T)+\frac{1}{2}g\circ gw^2(T)\Big)}_{F''_{\hat{v}(\cdot)}v(\cdot)}+R_3, \qquad (6.6)$$

where the remainder R_3 admits the following estimate for all controls $v(\cdot)$, satisfying, for example, $\|v(\cdot)\|_\infty \le 1$:

$$|R_3|\le C\Bigg(|w(T)|\underbrace{\Bigg(|w^2(T)|+|w(T)|\int_0^T|w(\tau_1)|\,d\tau_1+\int_0^T w^2(\tau_1)\,d\tau_1+}_{I_1(w(\cdot))}$$

$$\underbrace{+\int_0^T d\tau_1|w(\tau_1)|\int_0^{\tau_1}|w(\tau_2)|\,d\tau_2\Bigg)}_{I_1(w(\cdot))}+$$

$$\underbrace{+\int_0^T|w^3(\tau_1)|\,d\tau_1+\int_0^T d\tau_1|w(\tau_1)|\int_0^{\tau_1}w^2(\tau_2)\,d\tau_2+}_{I_2(w(\cdot))}$$

$$\underbrace{+\int_0^T d\tau_1 w^2(\tau_1)\int_0^{\tau_1}|w(\tau_2)|\,d\tau_2+}_{I_2(w(\cdot))}$$

$$\underbrace{+\int_0^T d\tau_1|w(\tau_1)|\int_0^{\tau_1}|w(\tau_2)|\,d\tau_2\int_0^{\tau_2}|w(\tau_3)|\,d\tau_3\Bigg)}_{I_2(w(\cdot))}=$$

$$=C\Big(|w(T)|I_1(w(\cdot))+I_2(w(\cdot))\Big)\le$$

$$\le C\Big(\|v(\cdot)\|_1 I_1(w(\cdot))+I_2(w(\cdot))\Big). \qquad (6.7)$$

Using consequently the inequality $|w(\tau)| \le \tau\|v(\cdot)\|_\infty$ and Cauchy–Schwartz inequality, one can estimate each term of I_1 and I_2 and obtain

$$I_1(w(\cdot)) \le C_2\|v(\cdot)\|_\infty\|v(\cdot)\|_1, \tag{6.9}$$

$$I_2(w(\cdot)) \le C_3\|v(\cdot)\|_\infty\|v(\cdot)\|_1^2. \tag{6.10}$$

For example,

$$\int_0^T |w(\tau_1)|\Big(\int_0^{\tau_1} |w(\tau_2)|\,d\tau_2\Big)\,d\tau_1| \le$$
$$\le \int_0^T |w(\tau_1)|\Big(\int_0^{\tau_1} \tau_2\|v(\cdot)\|_\infty\,d\tau_2\Big)\,d\tau_1 \le$$
$$\le \|v(\cdot)\|_\infty \int_0^T \frac{\tau_1^2}{2}|w(\tau_1)|\,d\tau_1 \le$$
$$\le \frac{T}{2}\|v(\cdot)\|_\infty \int_0^T \tau|w(\tau)|\,d\tau \le$$
$$\le \|v(\cdot)\|_\infty \frac{T^{\frac{3}{2}}}{2}\Big(\int_0^T \tau^2 w^2(\tau)\,d\tau\Big)^{\frac{1}{2}} \le$$
$$\le C\|v(\cdot)\|_\infty\Big(\int_0^T \tau w^2(\tau)\,d\tau\Big)^{\frac{1}{2}} \le C\|v(\cdot)\|_\infty\|v(\cdot)\|_1$$

or

$$\int_0^T d\tau_1|w(\tau_1)|\int_0^{\tau_1} w^2(\tau_2)\,d\tau_2 \le \int_0^T d\tau_1|w(\tau_1)|\int_0^{\tau_1} \tau_2\|v\|_\infty|w(\tau_2)|\,d\tau_2 \le$$
$$\le \|v\|_\infty \int_0^T d\tau_1|w(\tau_1)|\int_0^{\tau_1} \sqrt{\tau_1\tau_2}\,|w(\tau_2)|\,d\tau_2 \le$$
$$\le \|v\|_\infty \int_0^T d\tau_1\sqrt{\tau_1}\,|w(\tau_1)|\int_0^{\tau_1} \sqrt{\tau_2}\,|w(\tau_2)|\,d\tau_2 \le$$
$$\le \|v\|_\infty\Big(\int_0^T \sqrt{\tau}\,|w(\tau)|d\tau\Big)^2 \le$$

$$\leq T\|v\|_\infty \int_0^T \tau w^2(\tau)\,d\tau \leq T\|v(\cdot)\|_\infty \|v(\cdot)\|_1^2.$$

Using estimates (6.7)–(6.10) and expansion (6.6), one can easily obtain estimates (6.1) and (6.2).

To complete the proof, it remains to check that the quadratic form $\lambda(T)\cdot F''_{\hat{v}(\cdot)}$ is continuous in H. Define the following two quadratic forms

$$Q_1(v(\cdot)) = \lambda(T)\cdot F'_{\hat{v}(\cdot))}v(\cdot)\circ F'_{\hat{v}(\cdot))}v(\cdot), \quad Q_2 = \lambda(T)\cdot F''_{\hat{v}(\cdot)} - Q_1.$$

Prove that the form Q_2 is continuous even in H_1. From the expression for $F'_{\hat{v}(\cdot)}$ in (6.6) it follows that

$$\begin{aligned} Q_2(v(\cdot)) = \frac{1}{2}\lambda(T)\cdot \Bigg(&\int_0^T [\dot{Y}_\tau g, Y_\tau g](q_1) w^2(\tau)\,d\tau + \\ &+ \int_0^T d\tau\, w(\tau) \int_0^\tau [\dot{Y}_s g, \dot{Y}_\tau g](q_1) w(s)\,ds + \\ &+ w(T)\int_0^T [g, \dot{Y}_\tau g](q_1) w(\tau)\,d\tau\Bigg). \end{aligned}$$

Note that $\dot{Y}_\tau g = Y_\tau[f,g]$. Therefore,

$$\lambda(T)\cdot[\dot{Y}_\tau g, Y_\tau g](q_1) = \lambda(0)\cdot[[f,g],g](\gamma(\tau)) = P(\tau)$$

and $\lambda(T)\cdot[\dot{Y}_s g, \dot{Y}_\tau g](q_1) = K(\tau,s)$, where the functions $P(\tau)$ and $K(\tau,s)$ are the same as in Sec. 3. Denote by $R(\tau) = \lambda(T)\cdot[g,\dot{Y}_\tau g](q_1)$. By the assumption, the second variation is diagonalizable, i.e., $K(\tau,s)\equiv 0$. Hence the form Q_1 can be written as follows

$$2\,Q_2(v(\cdot)) = \int_0^T P(\tau)w^2(\tau)\,d\tau + w(T)\int_0^T R(\tau)w(\tau)\,d\tau.$$

Let us recall that $P(0)=0$. If we prove that $R(0)=0$, then the form Q_2 is obviously continuous in H_1. Let us prove that $R(0)=0$. Indeed, denote by $\phi(s) = \lambda(T)\cdot[Y_s g, \dot{Y}_0 g](q_1)$, then $\phi(0) = \lambda(T)\cdot[Y_0 g, \dot{Y}_0 g](q_1) = \lambda(0)\cdot[g,[f,g]](q_0) = P(0) = 0$ and $\phi'(s) = \lambda\cdot[\dot{Y}_s g, \dot{Y}_0 g](q_1) = K(s,0) = 0$ for all $s \Rightarrow \lambda(T)\cdot[g,\dot{Y}_0 g](q_1) = R(0) = 0$. Now we prove continuity of the form Q_1 in H. Denote by $Q_1(v,u) = \lambda(T)\cdot F'_{\hat{v}(\cdot))}v(\cdot)\circ F'_{\hat{v}(\cdot))}u(\cdot)$. Let $v = v_1 + v_2$, where $v_1 \in \operatorname{Ker} F'_{\hat{v}}$ and $v_2 \in Z$. Then

$$Q_1(v) = Q_1(v_1) + Q_1(v_1,v_2) + Q_1(v_2,v_1) + Q_1(v_2).$$

It is clear that $Q_1(v_1) = 0$ and $Q_1(v_1, v_2) = 0$. Since Z is finite dimensional, then the form $Q_3(v) = Q_1(v_2)$ is continuous in H.

Finally , let us show that the form $Q_4(v) = Q_1(v_2, v_1)$ is continuous even in H_1. Since $Q_1(v_1, v_2)$ vanishes, we can subtract it from $Q_1(v_2, v_1)$ to obtain commutators:

$$Q_4(v) = Q_1(v_2, v_1) = Q_1(v_2, v_1) - Q_1(v_1, v_2) = \frac{1}{2}\lambda \cdot q_1 \circ [F'_{\tilde{v}}v_2, F'_{\tilde{v}}v_1] =$$
$$= \frac{1}{2}\lambda \cdot q_1 \circ [g\, w_2(T) - \int_0^T \dot{Y}_\tau g\, w_2(\tau)d\tau, g\, w_1(T) - \int_0^T \dot{Y}_\tau g\, w_1(\tau)d\tau] =$$
$$= w_1(T)\int_0^T R(\tau)w_2(\tau)\, d\tau - w_2(T)\int_0^T R(\tau)w_1(\tau)\, d\tau,$$

where w_1 and w_2 are primitives of v_1 and v_2 (here we also use that $\lambda(T) \cdot [\dot{Y}_s g, \dot{Y}_\tau g](q_1) = K(\tau, s) \equiv 0$). The fact that $R(0) = 0$ implies again continuity of Q_4 in H_1. This completes the proof of the continuity of $\lambda(T) \cdot F''_{\tilde{v}(\cdot)}$ in H. The theorem is proved. □

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